Micro Analysis 2021

(1)

(a)

(i)

3 vectors in $\mathbb{R}^3$ if $ax + by + cz = 0$
only when $a = b = c = 0$
(lin. independent)

$$a\begin{pmatrix}1\\0\\3\end{pmatrix} + b\begin{pmatrix}-5\\3\\1\end{pmatrix} + c\begin{pmatrix}-1\\1\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & -5 & -1\\0 & 3 & 1\\3 & 1 & 2\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

notice $Aa = 0$ has one solution
iff A is invertible.

$$|A| = 1 \cdot \begin{vmatrix}3 & 1\\1 & 2\end{vmatrix} - (-5)\begin{vmatrix}0 & 1\\3 & 2\end{vmatrix} + -1 \cdot \begin{vmatrix}0 & 3\\3 & 1\end{vmatrix}$$

$$= 1(3\cdot2 - 1\cdot1) + 5(0-3) - 1(0-9)$$

$$= -1 \neq 0$$

$\therefore$ invertible $\Rightarrow$ non-singular

$\Rightarrow$ only one solution

$\therefore$ lin. independent

$\therefore$ basis

$$t = 1\begin{pmatrix}1\\0\\3\end{pmatrix} + -1\begin{pmatrix}-5\\3\\1\end{pmatrix} + 2\begin{pmatrix}-1\\1\\2\end{pmatrix}$$

$$t = \begin{pmatrix}1+5-2\\0-3+2\\3-1+4\end{pmatrix} \qquad t = \begin{pmatrix}4\\-1\\6\end{pmatrix}$$

in standard basis

$$t = 4\begin{pmatrix}1\\0\\0\end{pmatrix} + -1\begin{pmatrix}0\\1\\0\end{pmatrix} + 6\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\underline{\underline{t = \begin{pmatrix}4\\-1\\6\end{pmatrix}}}$$

(b)

(i)

$$a_0 + a_1 + a_2 + \dots + a_n = 0$$
$$a_0e^x + a_1e^x + a_2e^x + \dots + a_ne^x = 0$$
$$\dots \qquad \dots$$
$$a_0e^{nx} + a_1e^{nx} + a_2e^{nx} + \dots + a_ne^{nx} = 0$$

$$\begin{pmatrix}1 & 1 & 1 & \dots & 1\\ e^x & e^x & e^x & \dots & e^x\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ e^{nx} & e^{nx} & e^{nx} & \dots & e^{nx}\end{pmatrix}\begin{pmatrix}a_0\\a_1\\ \vdots\\a_n\end{pmatrix} = \begin{pmatrix}0\\0\\ \vdots\\0\end{pmatrix}$$

(b)

(i)

$$\alpha_0 + \alpha_1 e^{x} + \alpha_2 e^{2x} + \dots + a_n e^{nx} = 0$$

$x=0 \quad a_0 + a_1 + a_2 + \dots + a_n = 0$

$x=1 \quad a_0 + a_1 e + a_2 e^2 + \dots + a_n e^n = 0$

$x=2 \quad a_0 + a_1 e^2 + a_2 e^4 + \dots + a_n e^{2n} = 0$

$\vdots \quad \vdots \quad \vdots$

$x=n \quad a_0 + a_1 e^n + a_2 e^{2n} + \dots + a_n e^{n^2} = 0$

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e & e^2 & \cdots & e^n \\ 1 & e^2 & e^4 & \cdots & e^{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^n & e^{2n} & \cdots & e^{n^2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

lin. independent if $|A| \neq 0$

(Since $|A| \Rightarrow$ invertible $\Rightarrow$ one solution $(a_i = 0 \ \forall i)$)

$$|A| = 1 \cdot \begin{vmatrix} e & e^2 & \cdots & e^n \\ e^2 & e^4 & \cdots & e^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e^n & e^{2n} & \cdots & e^{n^2} \end{vmatrix} + 1 \begin{vmatrix} 1 & e^2 & \cdots & e^n \\ 1 & e^4 & & e^{2n} \\ \vdots & & \ddots & \vdots \\ 1 & e^{2n} & \cdots & e^{n^2} \end{vmatrix} + \dots +$$

(!) not sure how to show that this is true (!)

(ii)

$$\alpha_0 + \alpha_1 \ln x + \alpha_2 \ln 2x + \dots + \alpha_n \ln nx = 0$$

notice $\alpha_2 \ln 2x = \alpha_2 \ln 2 + \alpha_2 \ln x$

let $\alpha_0 = -\alpha_2 \ln 2 \qquad \alpha_1 = -\alpha_2$

$\therefore$ solutions when $a_i \neq 0 \ \exists i$

$\therefore$ not independent

(c)

(i)

f continuous at (a_1, a_2) hence

$\forall \varepsilon > 0 \quad \exists \delta > 0$ such that,

if $\|(x,y) - (a_1, a_2)\| < \delta$ then $\|f(x,y) - f(a_1,a_2)\| < \varepsilon$

$g_1(t) = f(t, a_2)$ continuous at a_1

if $\forall \varepsilon > 0 \quad \exists \delta > 0$ such that,

if $|x - a_1| < \delta$ then $|g_1(x) - g(a_1)| < \varepsilon$

$\cancel{= |f(a_1, a_2)}$

$= |f(x, a_2) - f(a_1, a_2)| < \varepsilon$

Consider same δ as for f.

$|x - a_1| = \|(x, a_2) - (a_1, a_2)\| < \delta$

hence

$|f(x, a_2) - f(a_1, a_2)| = |g_1(x) - g(a_1)| < \varepsilon$

(+ by symmetry for g_2).

(ii)

$g_1(x) = f(x, 0)$ continuous at $x = 0$ if

$\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t.

if $|x - 0| < \delta$ then $|\cancel{g(x)} - g_1(x) - g_1(0)| < \varepsilon$

$$g_1(x) = f(x,0) = \frac{0}{x^2+y^2} = 0$$

$$g_1(0) = 0$$

$|g_1(x) - g_1(0)| = |0| < \varepsilon \quad \forall \varepsilon > 0$ hence

continuous!

(same for $g_2(y)$).

$f(x,y)$ not continuous at $(0,0)$

Counter e.g.:

· find a sequence converging to $(0,0)$
s.t. ~~does not~~ $f(\{x_k\}_{k=1}^{\infty}, \{y_k\}_{k=1}^{\infty})$ does not converge to $f(0,0)$

$$f(0,0) = 0$$

$$\cancel{\left(\frac{1}{x}\right)^k \to 0} \quad \lim_{n\to\infty} \left\{\frac{1}{n^2}, \frac{1}{n}\right\}_{n=1}^{\infty} = (0,0)$$

$$f\left(\frac{1}{n^2}, \frac{1}{n}\right) = \frac{\frac{1}{n^4}}{\frac{1}{n^4} + \frac{1}{n^4}} = \frac{1}{2} \neq 0.$$

(2)

(a) concave objective & convex constraints.

Concavity / convexity of Lagrangian makes KT focs necessary + sufficient for global max. / min.

(b)

$$Df(x,y) = \begin{pmatrix} 4x - 2y & 4y - 2x - 9 \end{pmatrix}$$

$$D^2f(x,y) = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$$

$$|D^2f(x,y)| = 16 - 4 = 12 > 0$$

$$tr\left(D^2f(x,y)\right) = 8$$

$\therefore$ eigenvalues are both > 0

$\Rightarrow$ PD $\Rightarrow$ convex. (strictly)

$$Dg(x,y) = \begin{pmatrix} -8x & 1 \end{pmatrix}$$

$$D^2g(x,y) = \begin{pmatrix} -8 & 0 \\ 0 & 0 \end{pmatrix}$$

$|D^2g(x,y)| = 0$ eigenvalues $= -8$ and 0

$\Rightarrow$ NSD $\Rightarrow$ concave.

(c)

$$Df(x,y) = \begin{pmatrix} 4x-2y & 4y-2x-9 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

① $4x-2y=0$ ② $4y-2x=9$

① + 2x ②:

$$\cancel{4x}-2y+8y-\cancel{4x} = 0+18$$

$$6y = 18$$

$$\boxed{y=3 \quad x=\tfrac{3}{2}}$$

Strict convexity ⇒ these focs are sufficient for a global min.

(d)

Notice convex objective + concave (linear) constraints

$$h(x,y) = y - 4x^2 + 2$$

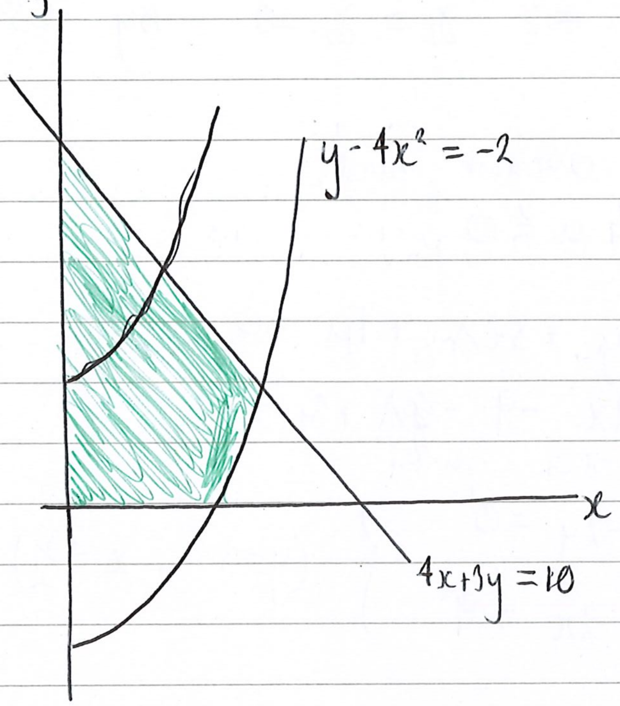
$$Dh(x,y) = \begin{pmatrix} -8x & 1 \end{pmatrix} \qquad D^2h(x,y) = \begin{pmatrix} -8 & 0 \\ 0 & 0 \end{pmatrix}$$

= concave.

hence Lagrangian is convex.

convex problem ⇒ KT focs are necessary + sufficient for a global minimum.

(i)



(ii)

$$\cancel{L = 2x^2 + 2y^2 - 2xy - 9y \mp \lambda(-2 + 4x^2 - y) \mp \mu(4x + 3y - 10) \mp \gamma_1(-x) - \gamma_2(-y)}$$

min. problem $\therefore$ $g(x) \geq 0$

$$L = 2x^2 + 2y^2 - 2xy - 9y - \lambda(y - 4x^2 + 2) - \mu(10 - 4x - 3y) - \gamma_1 x - \gamma_2 y$$

$$L_x = 4x - 2y + \lambda 8x + \mu 4 - \gamma_1 = 0$$

$$L_y = 4y - 2x - 9 - \lambda + 3\mu - \gamma_2 = 0$$

$\lambda \geq 0 \quad \lambda(y - 4x^2 + 2) = 0 \quad y - 4x^2 + 2 \geq 0$

$\mu \geq 0 \quad \mu(10 - 4x - 3y) = 0 \quad 10 - 4x - 3y \geq 0$

$\gamma_1 \geq 0 \quad \gamma_1 x = 0 \quad x \geq 0$

$\gamma_2 \geq 0 \quad \gamma_2 y = 0 \quad y \geq 0$

(iii) $x>0, \quad y>0 \implies \gamma_1 = \gamma_2 = 0$ by CS.

Case 1: neither constraint binds

$\implies \lambda = \mu = 0$

$$L_x = 4x - 2y + 8x\lambda + 4\mu = 0$$

$$L_y = 4y - 2x - 9 - \lambda + 3\mu = 0$$

$$\left.\begin{aligned} 4x - 2y &= 0 \\ 4y - 2x &= 9 \end{aligned}\right\} \left(y = 3, \quad x = \tfrac{3}{2}\right)$$

$$\cancel{4\times 3} \quad 4 \times \tfrac{3}{2} + 3 \times 3 = \underline{15 > 10}$$ $\therefore$ not a solution.

$\therefore$ at least one constraint binds

~~Case 2: both bind:~~

$$\cancel{4x + 3y = 10 \qquad y - 4x^2 = -2}$$

$$\cancel{y - 4\left(\tfrac{10}{4} - \tfrac{3}{4}y\right)^2 = -2}$$

$$\cancel{y - 4\left(\tfrac{10}{4} \times \tfrac{10}{4} + \tfrac{9}{16}y^2 - 2 \times \tfrac{3}{4} \times \tfrac{10}{4}y\right) = -2}$$

$$\cancel{y - 25 + \tfrac{9}{4}y^2 + 15y + 2 = 0}$$

$$\cancel{\tfrac{9}{4}y^2 + 16y - 23 = 0}$$

$$9y^2 + 64y - 92 = 0$$

Case 2: both bind

$$y - 4x^2 = -2 \qquad 4x + 3y = 10$$

$$y = \frac{10}{3} - \frac{4}{3}x$$

$$\frac{10}{3} - \frac{4}{3}x - 4x^2 = -2$$

$$0 = 4x^2 + \frac{4}{3}x - 2 - \frac{10}{3}$$

$$0 = 12x^2 + 4x - 16$$

$$0 = 3x^2 + x - 4$$

$$\boxed{\begin{matrix} x = 1 \\ y = 2 \end{matrix}} \quad (\text{or } -\tfrac{4}{3} \text{ but } x \geq 0!)$$

are $\lambda \geq 0 \quad \mu \geq 0$ met?

$$4(1) - 2(2) + 8(1)\cdot\lambda + 4\mu = 0 \qquad 4(2) - 2(1) - 9 - \lambda + 3\mu = 0$$

$$① \; 8\lambda + 4\mu = 0 \qquad ② \; -3 - \lambda + 3\mu = 0$$

① + 8 × ②:

$$\cancel{8\lambda} + 4\mu + 8(-3) + \cancel{8(-\lambda)} + 8(3\mu) = 0$$

$$\cancel{16\mu = 12} \quad 28\mu = 24$$

$$\mu = \frac{6}{7} \geq 0 \; ✓$$

$$\lambda = -\frac{3}{7} < 0 \; ✗$$

∴ not a solution

- intuitively given λ was neg given λ is the multiplier on $y \geq 4x^2 - 2$ then relaxing this constraint will $\uparrow$ the objective function at least <u>locally</u>

Generally:

Suppose $y = 4x^2 - 2$ $\quad 4x + 3y < 10$

$$\Rightarrow \mu = 0$$

$$4x - 2y + 8x\lambda = 0$$
$$4y - 2x - 9 - \lambda = 0$$

~~$4x - 2(4x^2 - 2) + 8x\lambda = 0$~~

$$\lambda = 16x^2 - 8 - 2x - 9$$

$$\lambda = 16x^2 - 2x - 17 < 0 \quad \forall x \text{ within the constraint set.}$$

$\therefore$ not a solution.

Case 4: $y - 4x^2 > -2 \quad \Rightarrow \lambda = 0 \quad 4x + 3y = 10$

① $4x + 3y = 10$

$$4x - 2y + 4\mu = 0$$
$$4y - 2x - 9 + 3\mu = 0$$

$$4x - 2y + 4\left(-\frac{4}{3}y + \frac{2}{3}x + \frac{9}{3}\right) = 0$$

② $-\ 5 \times$ ①:

$$4x - 2y + -\frac{16}{3}y + \frac{8}{3}x + 12 = 0$$

$$\cancel{20x} - 22y - \cancel{20x} - 15y = -36 - 50$$

$$-\frac{22}{3}y + \frac{20}{3}x = -12$$

$$-37y = -86$$

$$\boxed{y = \frac{86}{37}} \quad \boxed{x = \frac{28}{37}}$$

② $20x - 22y = -36$

$$\boxed{\mu = \frac{2}{4}y - x = \frac{15}{37} > 0} \quad ✓$$

~~Case 2:~~

(iv)

Convex problem hence global min.

(3)

$X = [0, 10]$

(a)

$$c(A) = \text{argmin}_{a_i} \left| a_i - \frac{\sum_{i=1}^{n} a_i}{n} \right|$$

$A \subseteq X$

$A \in \{a_1, a_2, \dots, a_n\}$

$$\text{mean } A = \frac{a_1 + a_2 + \dots + a_n}{n}$$

(b)

(i) for A, B s.t.

Prop. α : if for $B \subseteq A$, if $c(A) \in B$ then $c(B) = c(A)$

let $A = \{1, 2, 3, 9\}$ mean $A = 3.75$

$C(A) = 3$

let $B \subseteq A$ s.t. $B = \{1, 2, 3\}$ mean $B = 2$

$C(B) = 2$

but $B \subseteq A$ and $c(A) \in B$

$\therefore$ property α not satisfied.

(ii)

weak axiom : A & B cannot reveal $x \succ y$ and $y \succ x$

$A = \{1, 2, 3\}$ $c(A) = 2$ $\therefore$ $2 \succ 3$ $2 \succ 1$

$B = \{2, 3, 4\}$ $c(B) = 3$ $3 \succ 2$ $3 \succ 4$

breaks weak axiom.

(iii)

Consider $B = \{2, 3, 4\}$

$C(B) = 3$ which directly reveals that
$3 \succ 2$ and $3 \succ 4$.

Choice data is rationalizable iff it satisfies property α.

Only rational data admits of utility representation.
$\therefore$ no util. function that is guiding Mr Mean.

(c)

(i)

x is indirectly revealed as preferred to $y \neq x$ where there is a sequence of elements $x = z_1, z_2, \ldots, z_n = y$ such that $z_i = c(A_i)$ and $z_{i+1} \in A \setminus \{z_i\}$

that is $x \succ y$ where $x = z_i \succ z_2 \succ \ldots \succ z_n = y$

(ii)

$A = \{1, 3, 4\}$ $C(A) = 3$ directly: $3 \succ 1$ $3 \succ 4$

$B = \{2, 4, 6\}$ $C(B) = 4$ directly: $4 \succ 2$ $4 \succ 6$

let $g = 3$ and $g' = 6$.

$g = 3 \succ 4 \succ 6 = g'$

hence $\underline{\underline{g \succ g'}}$

(iii)

can only indirectly reveal by transitivity for Mr Mean hence we require a common element.

(note for a different c.f.) we could have indirectly revealed by monotonicity).

(d)

(i)

rational iff complete + transitive.

$C(A) = \underset{a_i}{\operatorname{argmin}} |a_i - 3|$
$A \in \{a_1, \dots, a_n\}$

Completeness: $\forall a, b$ $a \succsim b$, $b \succsim a$ or both

~~case 1:~~

① $|a-3| > |b-3| \Rightarrow b \succsim a$
② $|b-3| > |a-3| \Rightarrow a \succsim b$
③ $|b-3| = |a-3| \Rightarrow a \succsim b$ and $b \succsim a$

$\therefore$ complete

transitive:

if $a > b > 3$ and $c > a > 3$.

then $c > a > b > 3$

$\underline{\underline{c > a}}$

$\therefore$ rational.

Single-peakedness

if $a > b > 3$ then $b \succ a$ if $3 > c > d$ then $c \succ d$.

(ii)

$$C(A) = \underset{a_i}{\arg\min} \, |a_i - 3|$$
$$A \in \{a_1, \dots, a_n\}$$

$$u(a) = |a - 3|$$

$$a \succsim b \quad \text{iff} \quad u(a) = |a-3| \leq |b-3| = u(b)$$

(iii)

Peak rational, mean not.

(4)

(a)

(i)

$$\max \quad \tfrac{1}{2}u(W+Km) + \tfrac{1}{2}u(W-m) \qquad \text{s.t.} \quad K \geq 0, \quad m \in [0, W]$$

(ii)

foc :

$$\tfrac{1}{2}\frac{\partial}{\partial m}\mathbb{E}U = \cancel{\tfrac{1}{2}}\cdot K u'(W+Km) - \tfrac{1}{2}u'(W-m) = 0$$

$$\boxed{K = \frac{u'(W-m)}{u'(W+Km)}}$$

(iii)

$K=1 \implies W-m = W+Km \implies W-m = W+m \implies m=0.$

$\implies m=0$ since $u(x)$ is strictly increasing in x.

(risk averse).

$K<1 \implies u'(W-m) \not< u'(W+Km)$

$\cancel{u'(\cdot) < 0}$

Concavity of $u(\cdot) \implies u''(\cdot) < 0$

hence $u'(x)$ is decreasing in x.

$\therefore \; u'(W-m) \not< u'(W+Km)$ true when

$\cancel{W-m < W+Km}$ $\quad W-m > W+Km$

$\cancel{K>1 \; 1>K}$

$\cancel{K<1}.$

not possible for $\underline{K>0}$

$\therefore$ fails

Another argument:

② $(w-m, \ w+m)$ is a MPS of w ①

$\therefore$ ~~$w \succ$~~ ① $\succ$ ②

② FOSD ③ $(w-m, \ w+km)$ $k<1$ $\therefore$

① $\succ$ ③ by transitivity.

$K>1 \Rightarrow$ some investment.

$u'(w-m) > u'(w+km)$

given $u'(x)$ is decreasing in x.

(concavity $\Rightarrow$ $u''(x) < 0$)

this requires.

$w-m < w+km$

true for $k>1$

(b)

(i)

~~$K=$~~ $u(x) = x^{\frac{1}{2}}$ $\quad u'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$$K = \frac{\cancel{\frac{1}{2}}(w-m)^{-\frac{1}{2}}}{\cancel{\frac{1}{2}}(w-km)^{-\frac{1}{2}}} = \frac{\sqrt{w+km}}{\sqrt{w-m}}$$

$$\boxed{m^* = \frac{w(k-1)}{k}}$$

$$K(w-m)^{\frac{1}{2}} = (w+km)^{\frac{1}{2}}$$

$$k^2w - k^2m = w + km$$

$$w(k^2-1) = m(k^2+k)$$

$$m = \frac{w(k^2-1)}{(k^2+k)} = \frac{w(k^2-1)}{k(k+1)} = \frac{w\cancel{(k+1)}(k-1)}{k\cancel{(k+1)}}$$

$$\frac{\partial m^*}{\partial K} = w \frac{d}{dK}(K-1)K^{-1} = w\left[-K^{-2}(K-1) + K^{-1}(1)\right]$$

$$= wK^{-2}\left[K - K + 1\right]$$

$$= \cancel{\frac{w(2-K)}{K^2}} \quad \underline{\underline{\frac{w}{K^2}}} > 0 \quad \forall K.$$

$$\frac{\partial m^*}{\partial K} > 0$$

(ii)

$$u(CE) = EU \qquad u(x) = \sqrt{x} \qquad x = (u(x))^2$$

$$CE = u^{-1}(EU)$$

$$CE = \left(\tfrac{1}{2}u(w - m^*) + \tfrac{1}{2}u(w + Km^*)\right)^2$$

$$m^* = \frac{w(K-1)}{K}.$$

$$CE = \left[\tfrac{1}{2}u\left(w - \frac{w(K-1)}{K}\right) + \tfrac{1}{2}u\left(w + w(K-1)\right)\right]^2$$

$$= \left[\tfrac{1}{2}\left(w - \frac{w(K-1)}{K}\right)^{\frac{1}{2}} + \tfrac{1}{2}\left(w + w(K-1)\right)^{\frac{1}{2}}\right]^2$$

$$= \left[\tfrac{1}{2}\left(w\left(1 - \frac{K-1}{K}\right)\right)^{\frac{1}{2}} + \tfrac{1}{2}\left(w(1 + K - 1)\right)^{\frac{1}{2}}\right]^2$$

$$= \left[\tfrac{1}{2}\left(w\frac{1}{K}\right)^{\frac{1}{2}} + \tfrac{1}{2}(wK)^{\frac{1}{2}}\right]^2$$

Casey would accept: $\underline{\underline{\left[\tfrac{1}{2}\sqrt{\frac{w}{K}} + \tfrac{1}{2}\sqrt{wK}\right]^2 - w}}$

(c)

$$L'_{m_1} = \Big[p,\ 1-p\ ;\ \underbrace{\left[\tfrac{1}{2},\ \tfrac{1}{2},\ w-m_1,\ w+km_1\right]}_{L_{m_1}},\ w\Big]$$

$$L'_{m_2} = \Big[p,\ 1-p\ ;\ \underbrace{\left[\tfrac{1}{2},\ \tfrac{1}{2},\ w-m_2,\ w+km_2\right]}_{L_{m_2}},\ w\Big]$$

by independence $L_{m_1} \succsim L_{m_2}$ iff $L'_{m_1} \succsim L'_{m_2}$

hence $L_{m^*} \succsim L_{m'}$ $\forall m'$

iff

$L'_{m^*} \succsim L'_{m'}$ $\forall m'$

(d)

$$\mathbb{E}V = \int_{-m}^{km} (w+x)\frac{1}{(k+1)m}\,dx = \int_{w-m}^{w+km} x\frac{1}{(k+1)m}\,dx$$

$$\mathbb{E}U = \int_{-m}^{km} \sqrt{w+x}\,\frac{1}{(k+1)m}\,dx = \int_{w-m}^{w+km} \sqrt{x}\,\frac{1}{(k+1)m}\,dx$$

$$CE = \left[\int_{-m}^{km} \sqrt{w+x}\,\frac{1}{(k+1)m}\,dx\right]^2 = \left[\int_{w-m}^{w+km} \sqrt{x}\,\frac{1}{(k+1)m}\,dx\right]^2$$

(5)

(a)

$\max \ \pi = p - xk$ s.t. $\forall \theta \ \ V(m-p, x, \theta) \geq V(m, 0, \theta)$

$$\cancel{m} - p + 4\sqrt{\theta x} \geq \cancel{m}$$

$$4\sqrt{\theta x} \geq p.$$

- optimal for monopolist to $\uparrow p$ to $\uparrow \pi$ until $4\sqrt{\theta x} = p$.

$$\max_x \ \pi = 4\sqrt{\theta x} - xk.$$

$$0 = 4\theta^{1/2} \cdot \tfrac{1}{2} x^{-1/2} - k.$$

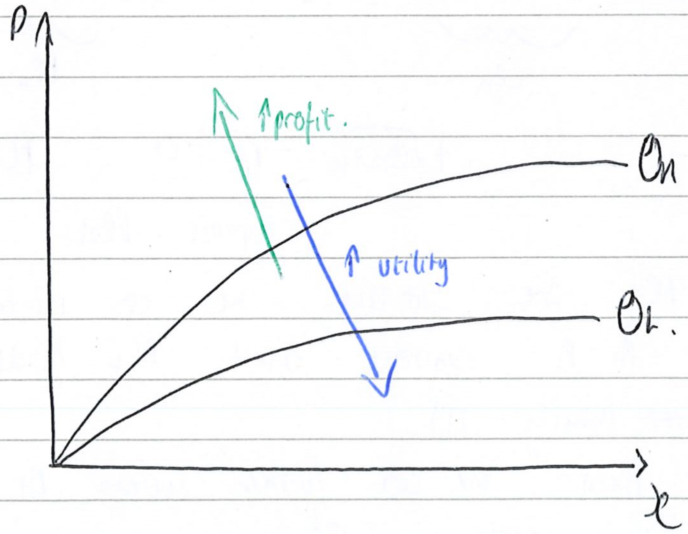
$$k = 2\sqrt{\theta} \frac{1}{\sqrt{x}}.$$

$$\sqrt{x^*} = \frac{2\sqrt{\theta}}{k} \qquad \boxed{x^* = \frac{4\theta}{k^2}} \qquad \boxed{p^* = \frac{8\theta}{k}}$$

(b)

Single crossing: $\dfrac{\partial^2 V}{\partial \theta \partial x} = \dfrac{1}{\sqrt{\theta x}} > 0$

$$\frac{\partial V}{\partial \theta} = 4 \tfrac{1}{2} \theta^{-1/2} x^{1/2} \qquad \frac{\partial^2 V}{\partial \theta \partial x} = 4 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} \frac{1}{\sqrt{\theta x}} = \frac{1}{\sqrt{\theta x}}$$



(c)

(i)

$$\max \quad \lambda(p_L - kx_L) + (1-\lambda)(p_H - kx_H)$$

subject to.

$$\boxed{PC_L: \quad m - p_L + 4\sqrt{\theta_L x_L} \geq m \qquad 4\sqrt{\theta_L x_L} - p_L \geq 0}$$

$$\boxed{PC_H: \quad 4\sqrt{\theta_H x_H} - p_H \geq 0}$$

$$IC_H: \quad \cancel{m} - p_H + 4\sqrt{\theta_H x_H} \geq \cancel{m} - p_L + 4\sqrt{\theta_H x_L}$$

$$IC_L: \quad \cancel{m} - p_L + 4\sqrt{\theta_L x_L} \geq \cancel{m} - p_H + 4\sqrt{\theta_L x_H}$$

$$\boxed{IC_H: \quad 4\sqrt{\theta_H x_H} - p_H \geq 4\sqrt{\theta_H x_L} - p_L}$$

$$\boxed{IC_L: \quad 4\sqrt{\theta_L x_L} - p_L \geq 4\sqrt{\theta_L x_H} - p_H.}$$

(ii)

Show that PC_L binds and IC_H binds.

- if IC_H and are satisfied so is PC_H.

$$4\sqrt{\theta_H x_H} - p_H \underbrace{\geq}_{IC_H} 4\sqrt{\theta_H x_L} - p_L \geq 4\sqrt{\theta_L x_L} - p_L \underbrace{\geq 0}_{PC_L}$$

$$4\sqrt{\theta_H x_H} - p_H \geq 0 \quad : PC_H$$

$\therefore$ ignore PC_H.

- if both IC's are satisfied we can increase p_L and p_H keeping $p_H - p_L$ constant until PC_L binds (~~incr~~ maxing Π)
- once p_L is fixed we can further increase p_H until IC_H binds (relaxing IC_L)

hence:

$$4\sqrt{\theta_L x_L} - P_L = 0 \qquad \boxed{P_L = 4\sqrt{\theta_L x_L}}$$

$$4\sqrt{\theta_H x_H} - P_H = 4\sqrt{\theta_H x_L} - P_L$$

$$P_H = 4\sqrt{\theta_H x_H} - 4\sqrt{\theta_H x_L} + 4\sqrt{\theta_L x_L}$$

(iii)

max. $\lambda(p_L - kx_L) + (1-\lambda)(p_H - kx_H)$

subject to

$$p_L = 4\sqrt{\theta_L x_L}$$
$$p_H = 4\sqrt{\theta_H x_H} - 4\sqrt{\theta_H x_L} + 4\sqrt{\theta_L x_L}$$

max. with L, or replace in objective:

$$\max \lambda\left(4\sqrt{\theta_L x_L} - kx_L\right) + (1-\lambda)\left(4\left(\sqrt{\theta_H x_H} - \sqrt{\theta_H x_L} + \sqrt{\theta_L x_L}\right) - kx_H\right)$$

$$foc_{x_L} = \lambda 4\sqrt{\theta_L} \cdot \tfrac{1}{2}x_L^{-1/2} - k\lambda + (1-\lambda)4(-\sqrt{\theta_H})\tfrac{1}{2}x_L^{-1/2} + (1-\lambda)4\sqrt{\theta_L}\tfrac{1}{2}x_L^{-1/2} = 0$$

$$\frac{2\lambda\sqrt{\theta_L}}{\sqrt{x_L}} - k\lambda - \frac{2(1-\lambda)\sqrt{\theta_H}}{\sqrt{x_L}} + \frac{2(1-\lambda)\sqrt{\theta_L}}{\sqrt{x_L}} = 0$$

$$2\lambda\sqrt{\theta_L} - k\lambda\sqrt{x_L} - 2(1-\lambda)\sqrt{\theta_H} + 2(1-\lambda)\sqrt{\theta_L} = 0$$

$$x_L = \frac{\left(2\lambda\sqrt{\theta_L} - 2(1-\lambda)\sqrt{\theta_H} + 2(1-\lambda)\sqrt{\theta_L}\right)^2}{(k\lambda)^2}$$

$$\boxed{x_L = \frac{\left(2\sqrt{\theta_L} - 2(1-\lambda)\sqrt{\theta_H}\right)^2}{k^2\lambda^2}}$$

$$x_L > 0 \implies \frac{2\sqrt{\theta_L} - 2(1-\lambda)\sqrt{\theta_H}}{\lambda k} > 0$$

$$\cancel{2}\sqrt{\theta_L} - \cancel{2}(1-\lambda)\sqrt{\theta_H} > 0$$

$$\sqrt{\theta_L} - \sqrt{\theta_H} > -\lambda\sqrt{\theta_H}$$

$$\boxed{\lambda > 1 - \frac{\sqrt{\theta_L}}{\sqrt{\theta_H}}}$$

$$\text{foc}_{x_H}: \quad (1-\lambda)\, 4\sqrt{\theta_H}\, \tfrac{1}{2}\, x_H^{-1/2} - (1-\lambda)k = 0$$

$$(1-\lambda)\left[\frac{2\sqrt{\theta_H}}{\sqrt{x_H}} - k\right] = 0$$

$$\sqrt{x_H} = \frac{2\sqrt{\theta_H}}{k}$$

$$\boxed{x_H = \frac{4\theta_H}{k^2}} = x_H^*$$

(d)

λ low then sell to θ_H at first best.

λ high enough then price discriminate.

Micro Analysis 2020

(1)

(a)

(i)

$$u_{n+1} = A u_n \qquad \begin{aligned} F_{n+2} &= F_{n+1} + F_n \\ F_{n+1} &= F_{n+1} \end{aligned}$$

$$\begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = A \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$$

$$\boxed{A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}$$

$$|A| = 0 - 1 = -1 \qquad \boxed{|A| = -1}$$

$$|A| \neq 0 \iff \underline{\underline{\text{invertible}}}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = -(1-\lambda)\lambda - 1 = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

$$\cancel{1-\lambda} \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\lambda = \frac{1+\sqrt{5}}{2}$$

$$x_1 = \frac{1+\sqrt{5}}{2} x_2 \qquad \boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}}$$

$$\lambda = \frac{1-\sqrt{5}}{2}$$

$$x_1 = \frac{1-\sqrt{5}}{2} x_2 \qquad \cancel{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix}} \qquad \boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}}$$

(ii)

$$\begin{matrix} F_{2020} \\ F_{2019} \end{matrix} = A \begin{matrix} F_{2019} \\ F_{2018} \end{matrix} = A^2 \begin{matrix} F_{2018} \\ F_{2017} \end{matrix} = A^3 \begin{matrix} F_{2017} \\ F_{2016} \end{matrix}$$

$$\begin{pmatrix} F_{2020} \\ F_{2019} \end{pmatrix} = A^i \begin{pmatrix} F_{2020-i} \\ F_{2019-i} \end{pmatrix}$$

for $i = 2018$

$$\boxed{\begin{pmatrix} F_{2020} \\ F_{2019} \end{pmatrix} = A^{2018} \begin{pmatrix} F_2 \\ F_1 \end{pmatrix}}$$

$$A^{2018} = V D^{2018} V^{-1}$$

$$V = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \qquad V^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad D^{2018} = \begin{pmatrix} \lambda_1^{2018} & 0 \\ 0 & \lambda_2^{2018} \end{pmatrix}$$

$$A^{2018} = \begin{pmatrix} \lambda_1^{2019} & \lambda_2^{2019} \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix} \frac{1}{\lambda_1 - \lambda_2}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1^{2019} - \lambda_2^{2019} & -\lambda_2 \lambda_1^{2019} + \lambda_1 \lambda_2^{2019} \\ \lambda_1 - \lambda_2 & -\lambda_1 \lambda_2 + \lambda_1 \lambda_2 \end{pmatrix}$$

$$F_{2020} = \frac{1}{\lambda_1 - \lambda_2} \left(\lambda_1^{2019} - \lambda_2^{2019} - \lambda_2 \lambda_1^{2019} + \lambda_1 \lambda_2^{2019} \right)$$

$$\lambda_1 = 1 - \lambda_2 \qquad \lambda_2 = 1 - \lambda_1$$

$$= \cancel{\frac{1}{1 - 2\lambda_2} \left((1-\lambda_2)^{2019} - \lambda_2 - \lambda_2 (1-\lambda_2)^{2019} + (1-\lambda_2) \lambda_2^{2019} \right)}$$

$$F_{2020} = \frac{1}{1-2\lambda_2}\left(\cancel{(1-\lambda_2)^{2019}} \lambda_1^{2019} - \lambda_2^{2019} - (1-\lambda_1)\lambda_1^{2019} + (1-\lambda_2)\lambda_2^{2019} \right)$$

$$= \frac{1}{1-2\lambda_2}\left(\cancel{\lambda_1^{2019}} - \cancel{\lambda_1^{2019}} + \lambda_1^{2020} - \cancel{\lambda_2^{2019}} + \cancel{\lambda_2^{2019}} - \lambda_2^{2020} \right)$$

$$F_{2020} = \frac{1}{1-2\lambda_2}\left(\lambda_1^{2020} - \lambda_2^{2020} \right)$$

$$\boxed{F_{2020} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{2020} - \left(\frac{1-\sqrt{5}}{2}\right)^{2020}}{\sqrt{5}}}$$

(b)

(i)

1st : is this a solution?

$\cancel{0 + ST + s^2}$

$0 + 0 + 0 + 0 = 0$ ✓

$0 + 0 + 1 - 1 = 0$ ✓

2nd is jacobian invertible at this point?

$$Df = \begin{pmatrix} 5x^4 + 2t & 2sy \\ 2y^2 + s^2 & 4xy \end{pmatrix} = \begin{pmatrix} 0+2 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$$

$$|Df| = 2 \cdot 0 - 0 \cdot 2 = 0 \iff \text{singular (non-invertible)}$$

hence ~~no~~ IFT does not apply.

(ii)

$$Df = \begin{pmatrix} 5x^4 + 2t & 2sy \\ 2y^2 + s^2 & 4xy \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 3 & 0 \end{pmatrix}$$

$$|Df| = 6 \neq 0 \quad \therefore \text{ invertible}$$

$$\therefore \text{ apply IFT}$$

$$\begin{pmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial s} \end{pmatrix} = - \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}^{-1} \begin{pmatrix} \frac{\partial f_1}{\partial s} \\ \frac{\partial f_2}{\partial s} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial f_1}{\partial s} \\ \frac{\partial f_2}{\partial s} \end{pmatrix} = \begin{pmatrix} y^2 + 2s \\ 2sx \end{pmatrix} = \begin{pmatrix} 1 + 2(-1) \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 \\ 3 & 0 \end{pmatrix}^{-1} = \frac{1}{6} \begin{pmatrix} 0 & 2 \\ -3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial s} \end{pmatrix} = -\frac{1}{6} \begin{pmatrix} 0 & 2 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\frac{1}{6} \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{pmatrix} = -\frac{1}{6} \begin{pmatrix} 0 & 2 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2x \\ 1 \end{pmatrix} = -\frac{1}{6} \begin{pmatrix} 0 & 2 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{6} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$$

$$\boxed{\frac{\partial x}{\partial s} = 0 \quad \frac{\partial y}{\partial s} = -\frac{1}{2} \quad \frac{\partial x}{\partial t} = -\frac{1}{3} \quad \frac{\partial y}{\partial t} = -\frac{1}{3}}$$

(2)

(a)

(i)

$$f(t\underline{x} + (1-t)\underline{y}) \geq tf(\underline{x}) + (1-t)f(\underline{y})$$

$\forall t \in (0,1)$ and $\forall \underline{x}, \underline{y} \in S$ where S is convex.

(ii)

$$\cancel{\min\{f,g\} =} \quad m(x) = \min\{f(x), g(x)\}$$

$$f(x) \geq m(x) \qquad g(x) \geq m(x)$$

$$f(\lambda a + (1-\lambda)b) \geq \lambda f(a) + (1-\lambda)f(b)$$
$$\geq \lambda m(a) + (1-\lambda)m(b)$$

$$g(\lambda a + (1-\lambda)b) \geq \lambda g(a) + (1-\lambda)g(a)$$
$$\geq \lambda m(a) + (1-\lambda)m(b)$$

at any given point $m(x) = f(x)$ or $= g(x)$

hence

$$m(\lambda a + (1-\lambda)b) \geq \lambda m(a) + (1-\lambda)m(b).$$

(iii)

$$Df(x,y) = \begin{pmatrix} \frac{1}{x} & \frac{3}{y} \end{pmatrix} \qquad D^2f(x,y) = \begin{pmatrix} -\frac{1}{x^2} & 0 \\ 0 & -\frac{3}{y^2} \end{pmatrix}$$

$$Dg(x,y) = \begin{pmatrix} y^3 & 3xy^2 \end{pmatrix} \qquad D^2f(x,y) = \begin{pmatrix} 0 & \cancel{6xy}\ 3y^2 \\ 3y^2 & 6xy \end{pmatrix}$$

$$Dh(x,y) = \begin{pmatrix} -2x+5y & -6y+5x \end{pmatrix} \qquad D^2h(x,y) = \begin{pmatrix} -2 & 5 \\ 5 & -6 \end{pmatrix}$$

$$|D^2 h(x,y)| = -2 \cdot -6 - 25 < 0$$

$\therefore$ eigenvalues have opposite sign

$\therefore$ ~~unde~~ ~~und~~ ~~neither~~ indefinite

$\boxed{\text{not concave}}$

$$|D^2 f(x,y)| = -\frac{1}{x^2} \cdot -\frac{3}{y^2} - 0 = \frac{3}{x^2y^2} > 0 \quad \forall x,y \neq 0$$

$$tr(D^2 f(x,y)) = -\frac{1}{x^2} - \frac{3}{y^2} < 0 \quad \forall x,y \neq 0$$

hence eigenvalues negative

$\Rightarrow$ ND $\Rightarrow$ ~~strictly~~ concave

$$|D^2 g(x,y)| = 0 - 9y^4 < 0 \quad \forall y$$

$\therefore$ eigenvalues have opposite sign

$\Rightarrow$ ~~neither~~ indefinite.

$\boxed{\text{not concave}}$

(b)

$u(x,y) = \ln x + 3\ln y$ s.t. $x + 4y \leq 20$ $4x + y \leq 20$

(i)

$u(x,y)$ is only ~~asymptot~~ tends to axis asymptotically that is as $x \to 0$ $MU_x \to \infty$ and as $y \to 0$ $MU_y \to \infty$ hence never choose $x = 0$ or $y = 0$.

and $\ln(z)$ is undefined for $z \leq 0$!!

(ii)

notice $u(x,y)$ is concave and linear constraints hence KT are necessary + sufficient for ~~optim~~ a global max.

$$L = \ln x + 3\ln y - \lambda(x + 4y - 20) - \mu(4x + y - 20)$$

$$L_x = \frac{1}{x} - \lambda - 4\mu = 0$$

$$L_y = \frac{3}{y} - 4\lambda - \mu = 0$$

$$\lambda \geq 0 \qquad \lambda(x + 4y - 20) = 0 \qquad x + 4y - 20 \leq 0$$

$$\mu \geq 0 \qquad \mu(4x + y - 20) = 0 \qquad 4x + y - 20 \leq 0$$

multipliers given the Δ in objective when constraint is relaxed

$\therefore$ shadow prices for capital (λ)
shadow price for labour (μ)

(iii)

Case ① : both bind :

① $x+4y = 20$ ② $4x+y=20$

① $- 4 \times$ ②

$$x - 4\times 4x + 4y - 4\times y = 20 - 4\times 20$$

$$-15x = -60$$

$$\boxed{x = 4 \quad , \quad y = 4}$$

Solution ✓

$$\boxed{u(4, 4) = 4\ln 4} = 5.545$$

① $\frac{1}{4} = \lambda + 4\mu$

② $\frac{3}{4} = 4\lambda + \mu$

①$-4\times$② : $-\frac{11}{4} = -15\lambda$

~~$\lambda > 0$~~ $\lambda = \frac{11}{60} > 0$ ✓

$\boxed{\mu = \frac{1}{60} > 0}$ ✓

case ② : $x+4y=20$ $4x+y<20$ $\Rightarrow$ $\mu = 0$

need $\lambda \geq 0$

$L_x = \frac{1}{x} = \lambda$ hence $\lambda \geq 0$ as required since $x>0$

$L_y = \frac{3}{y} = 4\lambda$ $\frac{3}{y} = \frac{4}{x}$ $3x = 4y$

$x+4y = 20$ $4x = 20$

$$\boxed{x = 5 \quad , \quad y = 3.75}$$

$4x+y > 20$ $\therefore$ not solution

$u(5, 3.75) = 5.57$

case ③ : $4x+y = 20$ $x+4y < 20$ $\Rightarrow$ $\lambda = 0$

$\frac{1}{x} = 4\mu$ $\frac{3}{y} = \mu$ $\frac{1}{x} = \frac{12}{y}$ $y = 12x$

$4x+y = 20$

$$\boxed{x = 1.25 \quad y = 15}$$ $u(1.25, 15) = 8.347$

~~$\mu >$~~ $\boxed{\mu > 0}$ ✓ but $x+4y > 20$

$\therefore$ not solution

case ④ : neither bind $\Rightarrow$ $\mu = 0$ $\lambda = 0$

$\frac{1}{x} = 0$ $\frac{3}{y} = 0$ No solution

(iv)

concave problem $\therefore$ neccessary & sufficient!

(v)

$u(x,y) = xy^3$ no longer concave but utility is ordinal
hence I would take logs

$$\log(u(x,y)) = \log x + 3\log y$$

and do the exact same analysis.

(3)

(a)

$L_1 \succ L_2$ iff $L_1 \succsim L_2$ and <u>not</u> $L_2 \succsim L_1$

~~$L_1 \succ L_2$ iff $p \cdot L_1 + (1-p) \cdot L \succsim p \cdot L_2 + (1-p) \cdot L$~~

and

not ~~$p \cdot L_1 + (1-p) L \precsim p \cdot L_2 + (1-p) \cdot L$~~

(i)

let independence : $L_1 \succsim L_2$ iff $L_1' \succsim L_2'$

1st direction

$L_1 \succ L_2 \Rightarrow L_1 \succsim L_2 \Rightarrow L_1' \succsim L_2'$

and not $L_1' \sim L_2'$ since

$L_1' \sim L_2' \Rightarrow L_2' \succsim L_1' \Rightarrow L_2 \succsim L_1$ which contradicts $L_1 \succ L_2$

hence $L_1 \succ L_2 \Rightarrow L_1' \succ L_2'$

2nd direction

$L_1' \succ L_2' \Rightarrow L_1' \succsim L_2' \Rightarrow L_1 \succ L_2$

hence $\boxed{L_1 \succ L_2 \text{ iff } L_1' \succ L_2'}$

(ii)

1st $L_1 \sim L_2 \Rightarrow L_1 \succsim L_2$ and $L_2 \sim L_1 \Rightarrow L_1' \succsim L_2'$ and $L_2' \succsim L_1'$

hence $L_1' \sim L_2'$

2nd $L_1' \sim L_2' \Rightarrow L_1 \succsim L_2$ and $L_2 \succsim L_1 \Rightarrow L_1 \sim L_2$

$$\boxed{L_1 \sim L_2 \text{ iff } L_1' \sim L_2'}$$

$L_1 \succsim L_2$ iff $p \cdot L_1 + (1-p) \cdot L_3 \succsim p \cdot L_2 + (1-p) L_3$

$L_3 \succsim L_4$ iff $q \cdot L_3 + (1-q) \cdot L_2 \succsim q \cdot L_4 + (1-q) L_2$

let ~~$p = (1-q)$~~ $q = (1-p)$

hence

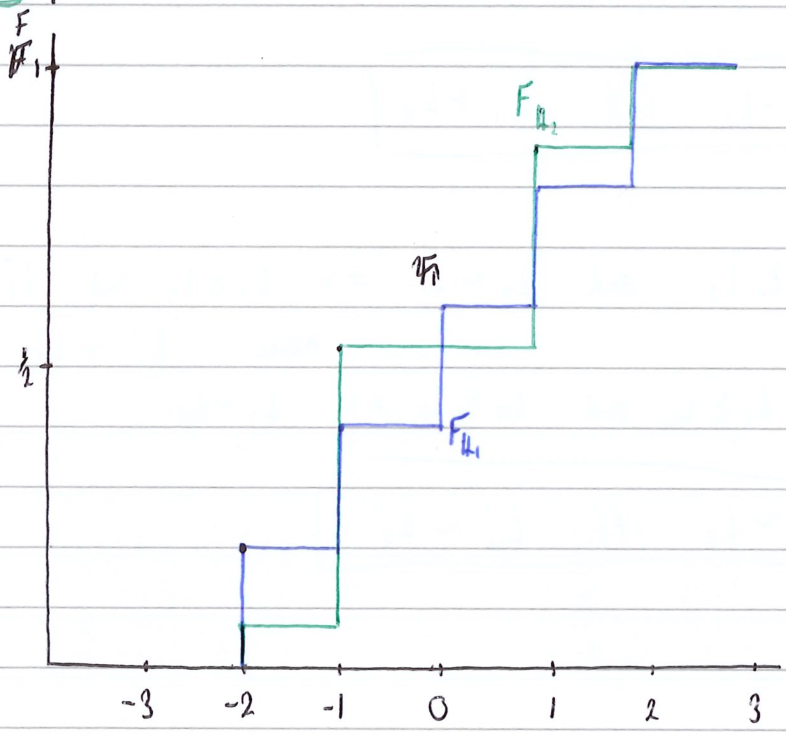
$L_3 \succsim L_4$ iff $(1-p) L_3 + p L_2 \succsim (1-p) L_4 + p L_2$

hence

$$\boxed{p \cdot L_1 + (1-p) L_3 \succsim (1-p) L_4 + p L_2}$$

(b)

	-3	-2	-1	0	1	+2
F_{L_1}	0	$1/5$	$2/5$	$3/5$	$4/5$	1
F_{L_2}	0	$1/15$	$8/15$	$8/15$	$13/15$	1



no FOSD or SOSD ?

(ii)

FOSD requires : $F_{4_1}(y) \leq F_{4_2}(y) \quad \forall y$

monotonicity requires : $p_1 \leq p_2 \leq p_3$

and $p_1 + p_2 + p_3 = 1$

notice cannot be that $p_1 > \frac{1}{3}$

hence $p_1 \leq \frac{1}{3}$

notice cannot be that $p_3 < \frac{1}{3}$

hence $p_3 \geq \frac{1}{3}$

	π_1	π_2	π_3
F_{4_1}	$\frac{1}{3}$	$\frac{2}{3}$	$\cancel{\frac{1}{3}}\ 1$
F_{4_2}	p_1	$p_1 + p_2$	$p_1 + p_2 + p_3 = 1$

FOSD requires $p_1 \leq \frac{1}{3}$ $\quad p_1 + p_2 \leq \frac{2}{3}$

$p_1 + p_2 = 1 - p_3$

$1 - p_3 \leq \frac{2}{3}$

$\boxed{p_3 \geq \frac{1}{3}}$

$\therefore$ monotonicity $\Rightarrow$ FOSD !

(4)

(a)

Principal max $\mathbb{E}[\pi - w \mid e]$ contracting e

such that

$$\sqrt{w} - e \geq 0$$

$$\mathbb{E}[\pi - w_L \mid e_0] = \tfrac{1}{4}4 + \tfrac{1}{2}\cdot 4 + \tfrac{1}{4}\cdot 40 - w_L = 13 - w_L$$

$$\mathbb{E}[\pi - w_H \mid e_1] = \tfrac{1}{4}\cdot 4 + \tfrac{1}{2}\cdot 40 + \tfrac{1}{4}\cdot 40 - w_H = 31 - w_H$$

Will offer w_L and w_H such that IR binds.

$$\sqrt{w_L} - 0 = 0$$

$$w_L = 0$$

$$\sqrt{w_H} - 2 = 0$$

$$w_H = 4$$

$$31 - 4 > 13 - 0$$

hence optimal contract:

$$\underline{\underline{(w_H, e_1) = (4, e_1)}}$$

(b)

induce e_0 by offering flat wage $\boxed{w_L = 0}$

such that $\sqrt{w_L} - 0 = 0$

induce e_1, w_L & w_H

IR: $\frac{1}{4} \overset{V_L}{\cancel{w_L}} + \frac{3}{4} V_H - 2 \geq 0$

IC: $\frac{1}{4} V_L + \frac{3}{4} V_H - 2 \geq \frac{3}{4} V_L + \frac{1}{4} V_H - 0$

<u>both will bind</u>

if IR binds then $0 \geq \frac{3}{4} V_L + \frac{1}{4} V_H$

(not possible)

hence only IC binds

$$\frac{1}{4} V_L + \frac{3}{4} V_H - 2 = \frac{3}{4} V_L + \frac{1}{4} V_H$$

$$\frac{1}{2} V_H - 2 = \frac{1}{2} V_L$$

$$V_H - V_L = 4$$

minimise wages by offering $V_L = 0$ $V_H = 4$

$w_L = 0$, $w_H = \cancel{4}\ 16$

$\mathbb{E}[w | e_H] = \frac{3}{4} 16 + \frac{1}{4} \times 0 = \underline{\underline{12}}$

$\mathbb{E}[\pi - w | e_H] = \frac{3}{4} 40 + \frac{1}{4} \cdot 4 - 12 = 19$

$19 > 13$ $\therefore$ optimal to offer

$(w_H, w_L) = (\cancel{12}, (16, 0)$

(c)

if machine were bad then

$\Pi = 4$ in either case hence

offer $w_L = 0$ that satisfies

IR constraint. and max. $[\Pi_L - w]$.

if machine is not bad:

	$\frac{1}{2} \to (\frac{2}{3})$	$\frac{1}{4} \to (\frac{1}{3})$
$\mid e_0$	4	40
$\mid e_1$	40	40

$$\mathbb{E}[\Pi \mid e_1] = 40 \qquad \mathbb{E}[\Pi \mid e_0] = \frac{2}{3} \cdot 4 + \frac{1}{3} 40 = 16.$$

to induce e_0 offer $w_L = 0$

to induce e_1 offer (w_L, w_H) s.t.

$$\cancel{\frac{2}{3} V_H + \frac{1}{3} V_L - 2} \geq 0$$

$$\cancel{\frac{2}{3} V_H + \frac{1}{3} V_L} - 2 \geq 1$$

$$V_H - 2 \geq 0$$

$$V_H - 2 \geq \frac{1}{3} V_H + \frac{2}{3} V_L - 0$$

again IC binds, IR does not.

$$V_H - 2 = \frac{1}{3} V_H + \frac{2}{3} V_L$$

$$\frac{2}{3} V_H - 2 = \frac{2}{3} V_L.$$

$$V_H - V_L = 3 \qquad V_L = 0 \qquad w_L = 0$$

$$V_H = 3 \qquad \boxed{w_H = 9}$$

$$\mathbb{E}[w \mid e_1] = 9$$

$$\mathbb{E}[\pi - w \mid e_1] = 40 - 9$$

$$31 > 16$$

$\therefore$ offer $(0, 3) = (w_L, w_H)$

s.t. induces $e_1 = e_H$

$$\mathbb{E}[\pi - w \mid \text{inspection}] = \frac{1}{4} \cdot 7 + \frac{3}{4} \cdot 31 = \frac{97}{4}$$

$$\mathbb{E}[\pi - w \mid \text{no inspection}] = 19$$

$$\frac{97}{4} - 19 = \frac{21}{4}$$

pay $< \frac{21}{4}$ for inspection

Micro Analysis 2019

(1)

(a)

(i)

$$\left| \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} (2-\lambda) & 1 & 1 \\ 1 & (2-\lambda) & 1 \\ 1 & 1 & (2-\lambda) \end{vmatrix} = 0$$

$$(2-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & (2-\lambda) \end{vmatrix} + 1 \begin{vmatrix} 1 & 2-\lambda \\ 1 & 1 \end{vmatrix} = 0$$

$$(2-\lambda)\left((2-\lambda)^2 - 1\right) - (1)\left((2-\lambda) - 1\right) + (1)\left(1 - (2-\lambda)\right) = 0$$

$$(2-\lambda)^3 - (2-\lambda) - (2-\lambda) + 1 + 1 - (2-\lambda) = 0$$

$$(2-\lambda)^3 - 3(2-\lambda) + 2 = 0$$

solution @ $\lambda = 1$, $\lambda = 4$

$$\begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

for $\lambda = 1 \quad \Rightarrow \quad x_1 + x_2 + x_3 = 0$

$\cancel{\text{for } \lambda = 4 \quad \Rightarrow \quad -2x_1 + x_2 + x_3 = 0}$

hence eigen vectors are ~~an~~ linear combinations of

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

for $\lambda = 4$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \underset{R_2 : R_2 + \frac{1}{2}R_1}{\overset{R_3 : R_3 + \frac{1}{2}R_1}{\Longrightarrow}} \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 0 \end{array}\right)$$

$$\underset{R_3 : R_3 - \frac{3}{2} R_1}{\Longrightarrow} \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & -3 & 0 \end{array}\right)$$

$$\underset{R_3 : R_3 + R_2}{\Longrightarrow} \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$-2x_1 + x_2 + x_3 = 0$$

$$-\tfrac{3}{2}x_2 + \tfrac{3}{2}x_3 = 0$$

$$x_2 = x_3.$$

$$-2x_1 + 2x_2 = 0 \qquad x_1 = x_2$$

$$\boxed{x_1 = x_2 = x_3}$$

$\therefore$ eigenvectors are multiples of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(ii)

$$V = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$A = V D V^{-1} \qquad V^{-1} = \frac{1}{3}\begin{pmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$A^6 = V D^6 V^{-1} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4096 \end{pmatrix}\begin{pmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix}\frac{1}{3}$$

$$= \begin{pmatrix} -1+0+0 & -1 & 4096 \\ 1 & 0 & 4096 \\ 0 & 1 & 4096 \end{pmatrix}\begin{pmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix}\frac{1}{3}$$

$$= \begin{pmatrix} 1 - 2 + 1 + 4096 & -2 + 1 + 4096 & 1 - 2 + 4096 \\ -1 + 4096 & 2 + 4096 & -1 + 4096 \\ -1 + 4096 & -1 + 4096 & 2 + 4096 \end{pmatrix}\frac{1}{3}$$

$$= \frac{1}{3}\begin{pmatrix} 4098 & 4095 & 4095 \\ 4095 & 4098 & 4095 \\ 4095 & 4095 & 4098 \end{pmatrix}$$

$$= \begin{pmatrix} 1366 & 1365 & 1365 \\ 1365 & 1366 & 1365 \\ 1365 & 1365 & 1366 \end{pmatrix}$$

(b) $\mathbb{R}^4$

1^{st} : basis $\longrightarrow$ span (u_1, u_2) intersection span (v_1, v_2) = null vector.

intersection : w $\quad w = au_1 + bu_2 \quad w = cv_1 + dv_2$

$$au_1 - cv_1 + bu_2 - dv_2 = 0$$

basis $\Rightarrow$ independence hence

$$a = -c = b = -d = 0$$

$\therefore$ basis implies intersection of span = null vector.

2^{nd} : intersect span $\rightarrow$ basis

basis : $\quad au_1 + bu_2 + cv_1 + dv_2 = 0$

?

(e)

(i)

$$f_x(0,y) = y\,\frac{0 - y^4 + 0}{0 + y^4} = y\,\frac{-y^4}{y^4} = -y$$

$$f_y(x,0) = x\,\frac{x^4 - 0 - 0}{x^4} = x$$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,0+h) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

$$\lim_{y \to 0} f_x(0,y) = 0 = f_x(0,0)$$

$$\lim_{x \to 0} f_y(x,0) = 0 = f_y(0,0)$$

$\therefore$ continuous at origin!

(ii)

$$f_{xx}(0,0) = \lim_{h\to 0} \frac{f_x(0+h,0) - f_x(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_{yy}(0,0) = \lim_{h\to 0} \frac{f_y(0,0+h) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,0+h) - f_x(0,0)}{h} = \lim_{h\to 0} \frac{-h-0}{h} = -1$$

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1$$

$$D^2 f(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

can't be continuous by youngs theorem!

(2)

(a)

f (strictly) concave $\Rightarrow$ local max. or stationary point $=$ (unique) global max.

concave objective & convex constraints $\Rightarrow$ KT focs are necessary and sufficient for global max.

need diff. to appeal to K-T ~~$\not\!$~~ foc's

(b)

~~$Df(x,y) = \begin{pmatrix} 2x - cy & 2y - cx \end{pmatrix}$~~

$$D^2 f(x,y) = \begin{pmatrix} 2 & -c \\ -c & 2 \end{pmatrix}$$

~~Det~~ $|D^2 f(x,y)| = 4 - c^2$

$tr(D^2 f(x,y)) = 4 > 0$

~~concave $\Leftarrow$ NSD convex $\Leftarrow$ PSD~~

~~for $4 > c$ matrix $f(x,y)$ is convex.~~

$c > 0$

for $4 - c^2 > 0 = 0 < c < 2$ then

$|D^2 f(x,y)| > 0$ $tr(D^2 f) > 0$ hence PD $\Rightarrow$ strictly convex.

for $c = 2$ then

$|D^2 f(x,y)| = 0$ $tr(D^2 f) > 0$ hence PSD $\Rightarrow$ weakly convex.

for $c > 2$ then

$|D^2 f(x,y)| < 0$ $tr(D^2 f) > 0$ ~~and~~ $\Rightarrow$ neither

(c)

(i)

$c = 1$ hence strictly convex

$\therefore$ stationary point = unique global min.

$$Df(x,y) = \begin{pmatrix} 2x - y & 2y - x \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

$$2x = y \qquad 2y = x.$$

$$\boxed{y = 0, \ x = 0} \qquad \boxed{f(0,0) = 0}$$

(ii)

No, $f(x,y)$ is strictly convex and in this domain is the global min.

hence $f(0,0)$ = only min.

(iii)

no longer convex set! and no longer contains $(0,0)$ hence yes answer must change.

$\cancel{f(\pm 4, y) = y^2 \pm 4y + 16}$

$f(4,y) = y^2 - 4y + 16 \qquad Df(4,y) = 2y - 4 \quad D^2f(4,y) = 2 > 0 \ \therefore$ convex.

$f(-4,y) = y^2 + 4y + 16 \qquad Df(-4,y) = 2y + 4 \quad D^2f(-4,y) = 2 > 0 \ \therefore$ convex.

$\therefore$ mins : $y = 2, \ y = -2.$

$f(x,4) = x^2 - 4x + 16$

$f(x,-4) = x^2 + 4x + 16$

$\therefore$ mins : $x = 2, \ x = -2$

new local minima at : $(4,2), (-4,-2), (2,4), (-2,-4)$

(d)

$$f(x,y) = x^2 + y^2 - xy \qquad s.t. \quad x^2 \leq 16 \qquad y^2 \leq 16$$

$$L = x^2 + y^2 - xy - \lambda(x^2 - 16) - \mu(y^2 - 16)$$

CQ:

~~if none bind~~ ~~then~~

$$Dg = \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}$$ full rank when one or both constraints bind (or ~~w~~ obviously when none do)

FOC's:

$$L_x = 2x - y - 2\lambda x = 0$$

$$L_y = 2y - x - 2\mu y = 0$$

$\lambda \geq 0 \qquad \lambda(x^2 - 16) = 0 \qquad x^2 - 16 \leq 0$

$\mu \geq 0 \qquad \mu(\cancel{x}y^2 - 16) = 0 \qquad y^2 - 16 \leq 0$

Case (1)

- none bind, hence $\mu = \lambda = 0$ and optima at $(0,0)$ (but a min.)

Case (2)

- both bind : $x^2 = 16$, $\cancel{x}y^2 = 16$ $\quad \lambda \geq 0$, $\mu \geq 0$

$x = 4$ or $x = -4$

$y = 4$ or $y = -4$

$(4,4)$	$(4,-4)$	$(-4,4)$	$(-4,-4)$	
$\mu = \frac{1}{2}$	$\mu = \frac{3}{2}$	$\mu = \frac{3}{2}$	$\mu = \frac{1}{2}$	≥ 0 $\therefore$ max!
$\lambda = \frac{1}{2}$	$\lambda = \frac{3}{2}$	$\lambda = \frac{3}{2}$	$\lambda = \frac{1}{2}$	

case (3): $x^2 = 16$ $\quad y^2 < 16$ $\quad \mu = 0$

$x = \pm 4$ $\quad L_y = 2y - x - \cancel{2\mu x}\,2\mu y = 0$

$2y = x$

$(4, 2)$ and $(-4, -2)$

$\lambda =$

$$\cancel{D^2 f(x,y)}\ BH = \text{Bordered hessian} = \begin{bmatrix} 0 & 2x & 0 \\ 2x & 2-2\lambda & -1 \\ 0 & -1 & 2-2\mu \end{bmatrix}$$

$$|BH| = 0 \cdot \begin{vmatrix} 2-2\lambda & -1 \\ -1 & 2-2\mu \end{vmatrix} - 2x \begin{vmatrix} 2x & -1 \\ 0 & 2-2\mu \end{vmatrix} + 0 \begin{vmatrix} 2x & 2-2\lambda \\ 0 & -1 \end{vmatrix}$$

$$|BH| = -2x\left(2x(2-2\mu) - 0\right)$$

$$= -4x^2(2-2\mu)$$

$\mu = 0$ $\quad |BH| = -8x^2 < 0$

$\therefore$ not max.

case (4): $y^2 = 16$ $\quad x^2 < 16$ $\quad \lambda = 0$

$y = \pm 4$ $\quad L_x = 2x - y - \cancel{2\lambda y}\,2\lambda x = 0$

$2x = y$

~~(4, 2) and (-4, -2)~~

$(2, 4)$ and $(-2, -4)$

$\mu = 3/4$ $\quad \mu = 3/4$

$$|BH| = -4x^2\left(2 - 2 \cdot \tfrac{3}{4}\right) = \underline{\underline{-2x^2}} < 0 \quad \therefore \text{not max.}$$

Overall :

max @ : $(4,4)$, $(4,-4)$, $(-4,4)$, $(-4,-4)$

(4)

(a)

$$\mathbb{L} = [1-\pi, \pi ; Y, Y-L]$$

$$\mathbb{L}_{insur.} = [1-\pi, \pi ; Y-\beta M, Y-L-\beta M+\beta L]$$

$$Y-\beta M = Y-M+(1-\beta)M$$

$$Y-L-\beta M+\beta L = Y-M+(1-\beta)M - (1-\beta)L$$

$$\max \quad (1-\pi)u(Y-M+(1-\beta)M) + \pi u(Y-M+(1-\beta)M-(1-\beta)L)$$

$$\max \quad (1-\pi)u(w+\alpha M) + \pi u(w+\alpha(M-L))$$

$$\equiv \max \quad V(\alpha) = \mathbb{E}[u(w+\alpha\tilde{x})]$$

(b)

α^* (& β^*) are determined by

foc: $\quad \mathbb{E}[\tilde{x}u'(w+\alpha^*\tilde{x})] = 0$

soc: $\quad \mathbb{E}[\underbrace{\tilde{x}^2}_{+ve}\underbrace{u''(w+\alpha\tilde{x})}_{-ve}] < 0 \quad \therefore \quad$ max.

if $\beta^* = 1$ then $\alpha^* = 0$

$$V'(0) = \mathbb{E}[\tilde{x}]u'(w) = [(1-\pi)M + \pi(M-L)]u'(w) > 0$$

$$= \underbrace{(M-\pi L)}_{+ve}\underbrace{u'(w)}_{+ve}$$

since $M > \pi L$ since $u(\cdot)$ was strictly increasing

$\therefore$ at $\alpha = 0$ $V'(\alpha) > 0$

$\therefore$ max is at $\alpha^* > 0 \Rightarrow \underline{\beta^* < 1}$

(c)

$$0 = \mathbb{E}[\tilde{x}\, u'(w + \alpha^* \cancel{w}\, \tilde{x})] = V'(\alpha^*) = V'(\alpha^*(w), w)$$

$$\frac{\partial}{\partial w} V'(\alpha^*(w), w) = \frac{\partial}{\partial \alpha} V' \cdot \frac{\partial \alpha}{\partial w} + \frac{\partial}{\partial w} V'$$

$$0 = \mathbb{E}[\tilde{x}^2 u''(w + \alpha^* \tilde{x})] \frac{\partial \alpha^*}{\partial w} + \mathbb{E}[\tilde{x}\, u''(w + \alpha^* \tilde{x})]$$

$$\frac{\partial \alpha^*}{\partial w} = - \frac{\mathbb{E}[\tilde{x}\, u''(w + \alpha^* \tilde{x})]}{\mathbb{E}[\tilde{x}^2 u''(w + \alpha^* \tilde{x})]} = \frac{-\mathbb{E}[\tilde{x}\, u''(w + \alpha^* \tilde{x})]}{\mathbb{E}[\tilde{x}^2 u''(w + \alpha^* \tilde{x})]}$$

denominator < 0 (see prev. part)
numerator $= ?$

if $A(y) = -\frac{u''(y)}{u'(y)}$ is decreasing in y then

$\forall$ non zero realisations x of $\tilde{x}$.

$$x\, A(w + \alpha^* x) < x\, A(w)$$

$$-x\, u''(w + \alpha^* x) < x\, A(w)\, u'(w + \alpha^* x)$$

$$-\mathbb{E}[\tilde{x}\, u''(w + \alpha^* \tilde{x})] < A(w)\, \underbrace{\mathbb{E}[\tilde{x}\, u'(w + \alpha^* \tilde{x})]}_{=0 \text{ by foc.}}$$

hence $-ve$

$$\Rightarrow \frac{\partial \alpha^*}{\partial w} = \frac{-ve}{-ve} = +ve > 0$$

$$\Rightarrow \frac{\partial \beta^*}{\partial w} < 0 \quad (\text{if } \text{DARA})$$

(5)

(a)

$$\max \ \mathbb{E}[\pi|\theta] \quad \text{s.t.} \quad w_H - g(e_H,\theta_H) \geq 0 \qquad w_L - g(e_L,\theta_L) \geq 0$$

$$\max \ \lambda\left[\pi(e_H) - w_H\right] + (1-\lambda)\left[\pi(e_L) - w_L\right] \quad \text{s.t.} \quad '' \qquad ''$$

$$L = \lambda\left[\pi(e_H) - w_H\right] + (1-\lambda)\left[\pi(e_L) - w_L\right] - \mu\left(g(e_H,\theta_H) - w_H\right)$$
$$- \gamma\left(g(e_L,\theta_L) - w_L\right)$$

$$L_{w_H} = -\lambda + \mu = 0 \qquad \mu = \lambda \geq 0$$

$$L_{w_L} = -(1-\lambda) + \gamma = 0 \qquad \gamma = (1-\lambda) \geq 0$$

hence by CS : $\cancel{\pi(e_L) - w_L = 0}$ $\boxed{\begin{aligned} w_L &= g(e_L,\theta_L) \\ w_H &= g(e_H,\theta_H) \end{aligned}}$

$$L_{e_H} = \lambda \pi'(e_H) - \mu g_e^H(e_H,\theta_H) = 0$$

$$L_{e_L} = (1-\lambda)\pi'(e_L) - \gamma g_e^L(e_L,\theta_L) = 0$$

$$\lambda \pi'(e_H) = \mu g_e^H(e_H,\theta_H)$$

$$\boxed{\pi'(e_H) = g_e^H(e_H,\theta_H)}$$

$$(1-\lambda)\pi'(e_L) = \gamma g_e(e_L,\theta_L)$$

$$\boxed{\pi'(e_L) = g_e(e_L,\theta_L)}$$

$$\pi'(e) = 2 \cdot \tfrac{1}{2} e^{-1/2} = e^{-1/2} \qquad \frac{1}{\sqrt{e}} = 1/\theta \qquad \boxed{\begin{aligned} \sqrt{e_H} &= \theta_H \\ \sqrt{e_L} &= \theta_L \end{aligned}}$$

$$g_e(e,\theta) = 1/\theta$$

hence

$$\boxed{\begin{aligned} w_L^* &= \frac{e_L}{\theta_L} = \theta_L \\ w_H^* &= \frac{e_H}{\theta_H} = \frac{\theta_H^2}{\theta_H} = \theta_H \end{aligned}}$$

Optimal contracts:

$$\boxed{\begin{aligned}(w_H^*, e_H^*) &= (\theta_H^*, \theta_H^2)\\ (w_L^*, e_L^*) &= (\theta_L, \theta_L^2)\end{aligned}}$$

(b)

(i)

Revelation Principle?

$$\max.\quad \lambda(\pi(e_H) - w_H) + (1-\lambda)(\pi(e_L) - w_L)$$

$$\text{s.t.}\ (PC_H)\quad w_H - g(e_H, \theta_H) \geq 0$$

$$(PC_L)\quad w_L - g(e_L, \theta_L) \geq 0$$

$$(IC_H)\quad w_H - g(e_H, \theta_H) \geq w_L - g(e_L, \theta_H)$$

$$(IC_L)\quad w_L - g(e_L, \theta_L) \geq w_H - g(e_H, \theta_L)$$

(ii)

if PC_L and IC_H then

$$w_L - g(e_L, \theta_L) \geq 0 \quad \text{and} \quad w_H - g(e_H, \theta_H) \geq w_L - g(e_L, \theta_H)$$

$g(e, \theta)$ is decreasing in θ $(g_\theta(e, \theta) < 0)$

hence given $\theta_H > \theta_L$ $g(e_L, \theta_H) < g(e_L, \theta_L)$

hence $w_L - g(e_L, \theta_L) < w_L - g(e_L, \theta_H)$

$\therefore$ $w_H - g(e_H, \theta_H) \geq w_L - g(e_L, \theta_H) \geq w_L - g(e_L, \theta_L) \geq 0$

hence $\boxed{w_H - g(e_H, \theta_H) \geq 0}$

Suppose $PC_L: \; w_L - g(e_L, \theta_L) > 0$

reduce w_L will $\uparrow \Pi$.

has not effect on IC_H

does impact IC_L : must reduce w_H by the same amount until,

$$\boxed{w_L - g(e_L, \theta_L) = 0}$$

Further reduce w_H until IC_H binds (to $\uparrow \Pi$)

$$\boxed{w_H - g(e_H, \theta_H) = w_L - g(e_L, \theta_H)}$$

$$\boxed{\begin{aligned} w_L &= g(e_L, \theta_L) \\ w_H &= g(e_L, \theta_L) - g(e_L, \theta_H) + g(e_H, \theta_H) \end{aligned}}$$

(iii)

Only need PC_L and IC_H (ignore IC_L)

$$L = \lambda(\Pi(e_H) - w_H) + (1-\lambda)(\Pi(e_L) - w_L) - \mu(w_L - g(e_L, \theta_L)) - \gamma(w_H - g(e_L, \theta_L) - g(e_H, \theta_H) + g(e_L, \theta_H))$$

① $L_{e_H} = \lambda \Pi'(e_H) + \gamma g_e(e_H, \theta_H) = 0$

② $L_{e_L} = (1-\lambda)\Pi'(e_L) + \mu g_e(e_L, \theta_L) + \gamma g_e(e_L, \theta_L) + - \gamma g_e(e_L, \theta_H) = 0$

$L_{w_H} = -\lambda - \cancel{\mu}\, \gamma = 0$ $\quad \lambda = -\gamma$

$L_{w_L} = -(1-\lambda) - \mu = 0$ $\quad (1-\lambda) = -\mu$

hence,

$$L_{e_H}: \quad \lambda \pi'(e_H) + \gamma g_e(e_H, \theta_H) = 0 \qquad (\lambda = -\gamma)$$

$$\cancel{\lambda} \pi'(e_H) = \cancel{\lambda} g_e(e_H, \theta_H)$$

$$\boxed{\pi'(e_H) = g_e(e_H, \theta_H)}$$

$$\hat{e}_H^{-1/2} = 1/\theta_H \qquad \boxed{\hat{e}_H = \theta_H^2} \qquad = e_H^* \; !$$

$$L_{e_L}: \quad (1-\lambda)\pi'(e_L) + \mu g_e(e_L, \theta_L) + \gamma g_e(e_L, \theta_{HL}) - \gamma g_e(e_L, \theta_H) = 0$$

$$(\lambda = -\gamma) \quad (-\mu = 1-\lambda)$$

$$(1-\lambda)\pi'(e_L) - (1-\lambda) g_e(e_L, \theta_L) - \lambda g_e(e_L, \theta_L) + \lambda g_e(e_L, \theta_H) = 0$$

$$\boxed{\pi'(e_L) = g_e(e_L, \theta_L) + \frac{\lambda}{1-\lambda}\left[g_e(e_L, \theta_L) - g_e(e_L, \theta_H)\right]}$$

$$\pi'(e) = e^{-1/2} \qquad g_e(e, \theta) = 1/\theta$$

$$\hat{e}_L^{-1/2} = 1/\theta_L + \frac{\lambda}{1-\lambda}\left[1/\theta_L - 1/\theta_H\right]$$

$$\theta_H > \theta_L \quad \text{hence} \quad 1/\theta_L > 1/\theta_H$$

$$1/\theta_L - 1/\theta_H > 0$$

$\therefore$

$$\hat{e}_L^{-1/2} > 1/\theta_L \quad \Leftrightarrow \quad \theta_L^2 > \hat{e}_L$$

$$e_L^* = \theta_L^2 \qquad e_L^* > \hat{e}_L$$

recall:

$$w_L = g(e_L, \theta_L)$$

$$w_H = g(e_L, \theta_L) - g(e_L, \theta_H) + g(e_H, \theta_H)$$

$$\hat{w}_L = \frac{\hat{e}_L}{\theta_L} \qquad \hat{e}_L < e_L^* \quad \text{hence} \quad \hat{w}_L < w_L^*$$

$$\hat{w}_H = \frac{\hat{e}_L}{\theta_L} - \frac{\hat{e}_L}{\theta_H} + \frac{\hat{e}_H}{\theta_H}$$

$$\hat{w}_H = \frac{\hat{e}_L}{\theta_L} - \frac{\hat{e}_L}{\theta_H} + \theta_H$$

$$\hat{w}_H = \theta_H + \hat{e}_L \underbrace{\left(1/\theta_L - 1/\theta_H\right)}_{+ve} > \theta_H = w_H^*$$

(iv)

$$g(\hat{e}_H, \theta_L) - g(\hat{e}_L, \theta_L) \geq g(\hat{e}_H, \theta_H) - g(\hat{e}_L, \theta_H)$$

$$\frac{\hat{e}_H}{\theta_L} - \frac{\hat{e}_L}{\theta_L} \geq \frac{\hat{e}_H}{\theta_H} - \frac{\hat{e}_L}{\theta_H}$$

$$\frac{\theta_H^2}{\theta_L} - \frac{\hat{e}_L}{\theta_L} \geq \frac{\theta_H^2}{\theta_H} - \frac{\hat{e}_L}{\theta_H}$$

$$\frac{1}{\theta_L}\cancel{\left(\theta_H^2 - \hat{e}_L\right)} \geq \frac{1}{\theta_H}\cancel{\left(\theta_H^2 - \hat{e}_L\right)}$$

$$\frac{1}{\theta_L} > 1/\theta_H \qquad !!$$