Econometrics 2021

(1)

(a)

$$\hat{u}_i = y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i$$

$$\hat{u}_i = (y_i - \bar{y}) + \hat{\beta}_2 \bar{x} - \hat{\beta}_2 x_i$$

$$= (y_i - \bar{y}) + \hat{\beta}_2 (\bar{x} - x_i)$$

$$y_i - \bar{y} = \cancel{\beta_1} + \beta_2 x_i + u_i - \cancel{\beta_1} - \beta_2 \bar{x} - \bar{u}$$

$$= (u_i - \bar{u}) + \beta_2 (x_i - \bar{x})$$

$$\hat{u}_i = (u_i - \bar{u}) + (x_i - \bar{x})(\beta_2 - \hat{\beta}_2)$$

$$= (u_i - \bar{u}) - (x_i - \bar{x})(\hat{\beta}_2 - \beta_2)$$

$$= (u_i - \bar{u}) - (x_i - \bar{x}) \frac{\sum (x_i - \bar{x})(u_i - \bar{u})}{\sum (x_i - \bar{x})^2}$$

$$\hat{u}_i^2 = (u_i - \bar{u})^2 + (x_i - \bar{x})^2 \frac{\left[\sum (x_i - \bar{x})(u_i - \bar{u})\right]^2}{\left[\sum (x_i - \bar{x})^2\right]^2} - 2 (x_i - \bar{x})(u_i - \bar{u}) \frac{\sum (x_i - \bar{x})(u_i - \bar{u})}{\sum (x_i - \bar{x})^2}$$

$$\sum \hat{u}_i^2 = \sum (u_i - \bar{u})^2 + \cancel{\sum (x_i - \bar{x})^2} \frac{\left[\sum (x - \bar{x})(u_i - \bar{u})\right]^2}{\left[\sum (x_i - \bar{x})^2\right]^{\cancel{2}}} - 2 \frac{\left[\sum (x_i - \bar{x})(u_i - \bar{u})\right]^2}{\sum (x_i - \bar{x})^2}$$

$$= \sum (u_i - \bar{u})^2 - \frac{\left[\sum (x_i - \bar{x})(u_i - \bar{u})\right]^2}{\sum (x_i - \bar{x})^2}$$

(b)

$$\hat{\sigma}^2 = \frac{1}{n-2}\sum(u_i-\bar{u})^2 - \frac{1}{n-2}\frac{\left[\sum(x_i-\bar{x})(u_i-\bar{u})\right]^2}{\sum(x_i-\bar{x})^2}$$

$$= \frac{n}{n-2}\underbrace{\frac{1}{n}\sum(u_i-\bar{u})^2}_{①} - \frac{n}{n-2}\frac{\frac{1}{n^2}\left[\sum(x_i-\bar{x})(u_i-\bar{u})\right]^2 \text{ ②}}{\frac{1}{n}\sum(x_i-\bar{x})^2 \text{ ③}}$$

$$\frac{n}{n-2} \xrightarrow{p} 1$$

① $\frac{1}{n}\sum(u_i-\bar{u})^2 = \frac{1}{n}\sum u_i^2 - \bar{u}^2$

$$\frac{1}{n}\sum u_i^2 \xrightarrow{p} \sigma_u^2 \quad \text{(iid LLN)}$$

$$\bar{u}^2 = \left[\frac{1}{n}\sum u_i\right]^2 \xrightarrow{p} \mathbb{E}[u_i]^2 = 0 \quad \text{(iid LLN)}$$

③ $\frac{1}{n}\sum(x_i-\bar{x})^2 = \frac{1}{n}\sum x_i^2 - \bar{x}^2$

$$\frac{1}{n}\sum x_i^2 \xrightarrow{p} \mathbb{E}[x_i^2] = \sigma_x^2$$

$$\bar{x}^2 = \left[\frac{1}{n}\sum x_i\right]^2 \xrightarrow{p} \mathbb{E}[x_i] = 0$$

② $\frac{1}{n^2}\left[\sum(x_i-\bar{x})(u_i-\bar{u})\right]^2 = \left[\frac{1}{n}\sum(x_i-\bar{x})(u_i-\bar{u})\right]^2$

$$= \left[\frac{1}{n}\sum(x_iu_i - \bar{x}u_i)\right]^2$$

Consider

$\frac{1}{n}\sum x_iu_i \qquad \mathbb{E}[x_iu_i] = \mathbb{E}[x_i]\,\mathbb{E}[u_i] = 0$

$$\frac{1}{n}\sum x_iu_i \xrightarrow{p} 0 \qquad \text{(iid LLN).}$$

$\frac{1}{n}\bar{x}\sum u_i \qquad \frac{1}{n}\sum u_i \xrightarrow{p} 0 \qquad \therefore \frac{1}{n}\bar{x}\sum u_i \xrightarrow{p} 0$

$\qquad\qquad \frac{1}{n}\sum x_i \xrightarrow{p} 0$

Overall:

$$\hat{\sigma}^2 \xrightarrow{p} (1)\cdot \sigma_u^2 - (1)\cdot \frac{[0-0]^2}{\sigma_x^2 - 0}$$

$$\hat{\sigma}^2 \xrightarrow{p} \sigma_u^2 \quad \therefore \quad \text{consistent.}$$

(c)

$$\hat{\sigma}^2 = \frac{n}{n-2}\left[\frac{1}{n}\sum u_i^2 + \bar{u}^2\right] - \frac{n}{n-2}\,\frac{\left[\frac{1}{n}\sum(x_i-\bar{x})(u_i-\bar{u})\right]^2}{\frac{1}{n}\sum(x_i-\bar{x})^2}$$

$$\hat{\sigma}^2 - \sigma^2 = \frac{n}{n-2}\left[\frac{1}{n}\sum(u_i^2-\sigma_u^2) + \sigma_u^2 + \bar{u}^2\right] - \cdots$$

1

$$\sqrt{n}(\hat{\sigma}^2 - \sigma^2) =$$

1

$$= \frac{n}{n-2}\, n^{-1/2}\sum(u_i^2-\sigma^2) + \sqrt{n}\frac{n}{n-2}\sigma^2 + \frac{n\sqrt{n}}{n-2}\bar{u}^2 - \sqrt{n}\sigma^2 - \sqrt{n}\frac{n}{n-2}\,\frac{\left[\frac{1}{n}\sum\cdots\right]^2}{\frac{1}{n}\sum(x_i-\bar{x})^2}$$

$$\frac{n}{n-2} \xrightarrow{p} 1$$

$$n^{-1/2}\sum(u_i^2-\sigma^2) \xrightarrow{D} N(0, \operatorname{var}(u_i^2))$$

$$\sqrt{n}\frac{n}{n-2}\sigma^2 \xrightarrow{p} 0$$

$$\frac{n}{n-2}\sqrt{n}\,\bar{u}^2 \xrightarrow{p} 0$$

$$-\sqrt{n}\,\sigma^2 \xrightarrow{p} 0$$

$$\frac{1}{n}\sum(x_i-\bar{x})^2 \xrightarrow{p} \sigma_x^2$$ (see previous, iid LLN)

$$\left[\frac{1}{n}\sum(x_i-\bar{x})(u_i-\bar{u})\right]^2 \xrightarrow{p} 0$$ (see prev., iid LLN)

$$\sqrt{n} \xrightarrow{p} 0$$

$$\frac{n}{n-2} \xrightarrow{p} 1$$

$$\sqrt{n}\left(\hat{\sigma}^2-\sigma^2\right) = \frac{n}{n-2}\, n^{-1/2} \sum (u_i^2-\sigma^2) + \sqrt{n}\,\frac{n}{n-2}\,\sigma^2 - \sqrt{n}\,\frac{n}{n-2}\,\bar{u}^2$$

$$-\sqrt{n}\,\sigma^2$$

$$-\sqrt{n}\,\frac{n}{n-2}\,\frac{\left[\frac{1}{n}\sum(x_i-\bar{x})(u_i-\bar{u})\right]^2}{\sum(x_i-\bar{x})^2}$$

$$\sqrt{n}\left(\hat{\sigma}^2-\sigma^2\right) \xrightarrow{D} N(0,\ var(u_i^2)) + 0 - 0 - 0 + 0\cdot 1\,\frac{(0)^2}{\sigma_x^2}$$

$$\xrightarrow{D} N(0,\ var(u_i^2))$$

(d)

$H_0:\ \sigma_u^2 = 1 \qquad H_1:\ \sigma^2 \neq 1$

$$t = \frac{\hat{\sigma}^2 - 1}{se(\hat{\sigma}^2)} = \frac{(\hat{\sigma}^2-1)}{\frac{\sqrt{var(u_i^2)}}{\sqrt{n}}} = \frac{\sqrt{n}\,(\hat{\sigma}^2-1)}{\sqrt{var(u_i^2)}}$$

$$\sqrt{n}\,(\hat{\sigma}^2-1) \xrightarrow{D} N(0,\ var(u_i^2))$$

$$t \xrightarrow{D} \frac{N(0,\ var(u_i^2))}{\sqrt{var(u_i^2)}} = N(0,1) \quad \text{under } H_0$$

at 5% sig. reject H_0 if $\underline{\underline{|t| > CV_{0.05} = 1.96}}$

(2)

(a)

$$y_i = \mu_1 1_{(i \le n_1)} + \mu_2 1_{(i > n_1)} + u_i$$

$$y_i = \begin{cases} \mu_1 + u_i & \text{if } i \le n_1 \\ \mu_2 + u_i & \text{if } i > n_1 \end{cases}$$

$$\mathbb{E}[y_i] = \begin{cases} \mu_1 & \text{if } i \le n_1 \\ \mu_2 & \text{if } i > n_1 \end{cases}$$

μ_1 and μ_2 are expected values of y_i for $i \le n_1$ and $i > n_1$ respectively.

(b)

$$\text{argmin} \sum (y_i - \mu_1 1_{i \le n_1} - \mu_2 1_{i > n_1})^2$$

focs

① $$\cancel{-2} \sum_{i=1}^{n} 1_{i \le n_1} (y_i - \hat{\mu}_1 1_{i \le n_1} - \hat{\mu}_2 1_{i > n_1}) = 0$$

② $$\cancel{-2} \sum_{i=1}^{n} 1_{i > n_1} (y_i - \hat{\mu}_1 1_{i \le n_1} - \hat{\mu}_2 1_{i > n_1}) = 0$$

notice $1_{i \le n_1} \cdot 1_{i > n_1} = 0$

①:

$$\sum_{i=1}^{n} 1_{i \le n_1} y_i - \hat{\mu}_1 \sum_{i=1}^{n} (1_{i \le n_1})^2 = 0$$

$$\hat{\mu}_1 = \frac{\sum_{i=1}^{n} 1_{i \le n_1} y_i}{\sum_{i=1}^{n} (1_{i \le n_1})^2} = \frac{\sum_{i=1}^{n_1} y_i}{n_1} = \boxed{\frac{1}{n_1} \sum_{i=1}^{n_1} y_i}$$

②

$$\hat{\mu}_2 = \frac{\sum_{i=1}^{n} 1_{i > n_1} y_i}{\sum_{i=1}^{n} (1_{i > n_1})^2} = \frac{\sum_{i=n_1+1}^{n} y_i}{n - n_1} = \boxed{\frac{1}{n - n_1} \sum_{i=n_1+1}^{n} y_i}$$

(c)

$$F = \frac{(R\hat{\beta} - q)' \left\{ \hat{\sigma}^2 R (X'X)^{-1} R' \right\}^{-1} (R\hat{\beta} - q)}{J}$$

$$H_0 : \mu_1 - \mu_2 = 0 \qquad H_1 : \mu_1 - \mu_2 \neq 0$$

$$R = \begin{pmatrix} 1 & -1 \end{pmatrix} \qquad \hat{\beta} = \begin{pmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \end{pmatrix} \qquad q = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad J = 1$$

$$F = \frac{\left[\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \end{pmatrix}\right]^T \left\{ \hat{\sigma}^2 \begin{pmatrix} 1 & -1 \end{pmatrix} (X'X)^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}^{-1} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \end{pmatrix}}{1}$$

$$\left(\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \end{pmatrix}\right)^T = (\hat{\mu}_1 - \hat{\mu}_2)^T = (\hat{\mu}_1 - \hat{\mu}_2)$$

$$F = \frac{(\hat{\mu}_1 - \hat{\mu}_2) \left\{ \hat{\sigma}^2 \begin{pmatrix} 1 & -1 \end{pmatrix} (X'X)^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}^{-1} (\hat{\mu}_1 - \hat{\mu}_2)}{}$$

$(X'X)^{-1}$?

$$Y = X\mu + u$$

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \qquad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \qquad \therefore \cancel{X = \begin{pmatrix} 1_{i \le n_1} \end{pmatrix}}$$ (it just has to!)

$$\cancel{\hat{\mu} = (X'X)^{-1} X'Y} \qquad \cancel{X'Y = \begin{pmatrix} 1_{i \le n_1} & 1_{i > n_1} \end{pmatrix}} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$\therefore \quad X = \begin{pmatrix} 1_{1 \le n_1} & 1_{1 > n_1} \\ \vdots & \\ 1_{n \le n_1} & 1_{n > n_1} \end{pmatrix}$$

$$X' = \begin{pmatrix} 1_{1 \leq n_1} & \cdots & 1_{n \leq n_1} \\ 1_{1 > n_1} & \cdots & 1_{n > n_1} \end{pmatrix}$$

$$\hat{\mu} = (X'X)^{-1} X'Y$$

$$X'Y = \begin{pmatrix} 1_{1 \leq n_1} & \cdots & 1_{n \leq n_1} \\ 1_{1 > n_1} & \cdots & 1_{n > n_1} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n_1} y_i \\ \sum_{i=n_1+1}^{n} y_i \end{pmatrix}$$

$\therefore$ it must be the case that:

$$(X'X)^{-1} = \begin{pmatrix} \frac{1}{n_1} & 0 \\ 0 & \frac{1}{n-n_1} \end{pmatrix}$$

$$F = (\hat{\mu}_1 - \hat{\mu}_2)\left(\sigma^2 \begin{pmatrix} \frac{1}{n_1} & \frac{-1}{n-n_1} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)^{-1} (\hat{\mu}_1 - \hat{\mu}_2)$$

$$= (\hat{\mu}_1 - \hat{\mu}_2)\left(\sigma^2\left(\frac{1}{n_1} + \frac{1}{n-n_1}\right)\right)^{-1} (\hat{\mu}_1 - \hat{\mu}_2)$$

$$\boxed{F = \frac{(\hat{\mu}_1 - \hat{\mu}_2)^2}{\sigma^2\left(\frac{1}{n_1} + \frac{1}{n-n_1}\right)}} = t^2 \qquad (F_{1,\infty}) = t^2 \; ?$$

(d)

$\mu_1 \neq \mu_2 \quad \therefore$ we are under H_1

hence $\mu_1 - \mu_2 = c \qquad c \neq 0$

$$F = \frac{((\hat{\mu}_1 - \hat{\mu}_2) - c + c)^2}{\sigma^2\left(\frac{1}{n_1} + \frac{1}{n-n_1}\right)}$$

$$\frac{1}{n_1} - \frac{1}{n+n_1} = \frac{n+n_1-n_1}{n_1(n-n_1)} = \frac{\cancel{n-2n_1}}{n_1(n-n_1)} \quad \frac{n}{n_1(n-n_1)}$$

$$F = \frac{((\hat{\mu}_1 - \hat{\mu}_2) - c + c)^2 \cdot n_1(n-n_1)}{\cancel{n-2n_1}\, n\sigma^2}$$

$$\cancel{n_1 = \frac{n}{2}}$$

$$F = n_1(n-n_1)\left[\frac{(\hat{\mu}_1 - \hat{\mu}_2 - c)^2}{n\sigma^2} + \frac{c^2}{n\sigma^2} + 2\frac{(\hat{\mu}_1 - \hat{\mu}_2 - c)c}{n\sigma^2}\right]$$

$$n_1 = \frac{n}{2} \qquad \frac{n}{2}\left(n - \frac{n}{2}\right) \qquad \frac{n^2}{2} - \frac{n}{2} = \frac{n^2-n}{2}$$

$$F = \frac{\frac{1}{2}(n^2-n)\,(\hat{\mu}_1 - \hat{\mu}_2 - c)^2}{n\sigma^2} + \frac{\frac{1}{2}(n^2-n)}{n\sigma^2}c^2 + \frac{n^2-n}{n\sigma^2}(\hat{\mu}_1 - \hat{\mu}_2 - c)c$$

1st term converges to a distribution.

$$\frac{\frac{1}{2}(n^2-n)}{n\sigma^2} \xrightarrow{P} \frac{n}{\sigma^2}$$

$$\frac{n}{\sigma^2}(\hat{\mu}_1 - \hat{\mu}_2 - c)^2 = \frac{1}{\sigma^2}\left[\sqrt{n}\,(\hat{\mu}_1 - \hat{\mu}_2 - c)\right]^2 \xrightarrow{D} \ldots$$

2nd term

$$\xrightarrow{P} \frac{n}{\sigma^2}c^2 \xrightarrow{P} \infty$$

3rd term

$$\xrightarrow{P} \frac{n}{\sigma^2}(\hat{\mu}_1 - \hat{\mu}_2 - c)c = \sqrt{n}\frac{1}{\sigma^2}\frac{\sqrt{n}}{\sqrt{n}}(\hat{\mu}_1 - \hat{\mu}_2 - c)\cancel{\frac{1}{\sqrt{n}}} \xrightarrow{P} \infty \cdot \text{Distribution}$$

- Hence F stat explodes to ∞ under H_1
  - useful since $F \xrightarrow{p} \infty$ means that under H_1 $F > CV_\alpha$ $\Rightarrow$ we correctly reject the null.

(3)

(a)

$$\text{argmin} \sum (y_i - \beta)^2$$

foc:

$$-2 \sum (y_i - \hat{\beta}) = 0$$

$$\underline{\hat{\beta} = \frac{1}{n} \sum_{i=1}^{n} y_i} = \frac{1}{n} \sum_{i=1}^{n} (\beta + u_i) = \beta + \frac{1}{n} \sum_{i=1}^{n} u_i$$

$$\mathbb{E}[\hat{\beta}] = \beta + \frac{1}{n} \sum_{i=1}^{n} \underbrace{\mathbb{E}[u_i]}_{=0} \quad (\text{iid } N(0, (1/2)^i))$$

$$\mathbb{E}[\hat{\beta}] = \beta$$

$$\text{var}(\hat{\beta}) = \underbrace{\text{var}(\beta)}_{=0} + \frac{1}{n^2} \sum_{i=1}^{n} \text{var}(u_i) \qquad (\text{cov}(u_i, u_j) = 0 \ \forall i \neq j \text{ by } u_i \text{ iid})$$

$$\text{var}(\hat{\beta}) = \frac{1}{n^2} \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} \right]$$

$$= \frac{1}{n^2} \left[1 - \left(\frac{1}{2}\right)^n \right]$$

- Mean Square convergence implies convergence in probability

$$\mathbb{E}[\hat{\beta}] = \beta \qquad \text{Var}(\hat{\beta}) = \frac{1}{n^2} \left[1 - \left(\frac{1}{2}\right)^n \right]$$

$$\left. \begin{array}{l} \frac{1}{n^2} \xrightarrow{p} 0 \\ 1 - \left(\frac{1}{2}\right)^n \xrightarrow{p} 1 \end{array} \right\} \Rightarrow \text{var}(\hat{\beta}) \xrightarrow{p} 0$$

hence by MSC $\hat{\beta} \xrightarrow{p} \beta$

$\therefore$ consistent.

(b)

$$\hat{\beta} - \beta = \frac{1}{n} \sum_{i=1}^{n} u_i$$

$$\sqrt{n}(\hat{\beta} - \beta) = n^{-1/2} \sum_{i=1}^{n} (u_i - 0) \xrightarrow{D} N(0, \mathrm{var}(u_i))$$

$$\boxed{\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{D} N\left(0, \frac{1}{n^2}\left[1 - \left(\frac{1}{2}\right)^n\right]\right)}$$

wrong! X

$$\hat{\beta} \sim N\left(\beta, \frac{1}{n^2}\left[1 - \left(\frac{1}{2}\right)^n\right]\right)$$

$$\frac{\hat{\beta} - \beta}{\sqrt{\frac{1}{n}\left[1 - \frac{1}{2^n}\right]}} \sim N(0, 1)$$

$$\boxed{\begin{gathered} n(\hat{\beta} - \beta) \sim N\left(0, 1 - \frac{1}{2^n}\right) \\ n(\hat{\beta} - \beta) \xrightarrow{D} N(0, 1) \end{gathered}}$$

(c)

$$\frac{y_i}{\sqrt{1/2^i}} = \frac{\beta}{\sqrt{1/2^i}} + \tilde{u}_i$$

$$\tilde{\beta} = \operatorname{argmin}_{\beta} \sum_{i=1}^{n} \left(\frac{y_i}{\sqrt{1/2^i}} - \frac{\beta}{\sqrt{1/2^i}} \right)^2$$

foc:

$$\cancel{-2}\sum_{i=1}^{n} \frac{1}{\sqrt{1/2^i}} \left(\frac{y_i}{\sqrt{1/2^i}} - \frac{\tilde{\beta}}{\sqrt{1/2^i}} \right) = 0$$

$$\sum_{i=1}^{n} \frac{y_i}{1/2^i} - \tilde{\beta} \sum_{i=1}^{n} \frac{1}{1/2^i} = 0$$

(recall $y_i = \beta + u_i$)

$$\tilde{\beta} = \frac{\sum_{i=1}^{n} 2^i y_i}{\sum_{i=1}^{n} 2^i} = \beta + \frac{\sum_{i=1}^{n} 2^i u_i}{\sum_{i=1}^{n} 2^i}$$

(d)

$$\boxed{\mathbb{E}[\tilde{\beta}] = \beta + \frac{\sum_{i=1}^{n} 2^i \,\mathbb{E}[u_i]}{\sum_{i=1}^{n} 2^i} = \beta}$$

$$\operatorname{Var}(\tilde{\beta}) = \left[\frac{1}{\sum_{i=1}^{n} 2^i}\right]^2 \sum_{i=1}^{n} 2^{2i} \underbrace{\operatorname{var}(u_i)}_{= 1/2^i} = \left\{\frac{1}{\sum_{i=1}^{n} 2^i}\right\}^2 \sum_{i=1}^{n} 2^{2i} \, 1/2^i$$

$$\operatorname{Var}(\tilde{\beta}) = \frac{1}{\left[\sum_{i=1}^{n} 2^i\right]^2} \sum_{i=1}^{n} 2^{2i} \cdot 1/2^i = \frac{1}{\left[\sum_{i=1}^{n} 2^i\right]^{\cancel{2}}} \cancel{\sum_{i=1}^{n} 2^i} = \frac{1}{\sum_{i=1}^{n} 2^i} = \frac{1}{-2(1-2^n)}$$

$$\boxed{\operatorname{Var}(\tilde{\beta}) = \frac{1}{2(2^n-1)}}$$

$\tilde{\beta}$ more efficient since $\hat{\beta}$ has ~~homosk~~ heteroskedastic errors (Gauss-Markov)

(4)

(a)

$$f_{\beta_1,\beta_2}(y_i|x_i) = \left(\frac{\exp(B_i)}{1+\exp(B_i)}\right)^{y_i} \left(\frac{1}{1+\exp(B_i)}\right)^{1-y_i}$$

$$L_{y_1,\ldots,y_n|x_1,\ldots,x_n}(\beta_1,\beta_2) = \prod_{i=1}^{n} \left(\frac{\exp(B_i)}{1+\exp(B_i)}\right)^{y_i} \left(\frac{1}{1+\exp(B_i)}\right)^{1-y_i}$$

$$l_{y_1,\ldots,y_n|x_1,\ldots,x_n}(\beta_1,\beta_2) = \sum_{i=1}^{n} \ln\left(\left(\frac{\exp(B_i)}{1+\exp(B_i)}\right)^{y_i}\left(\frac{1}{1+\exp(B_i)}\right)^{1-y_i}\right)$$

$$= \sum_{i=1}^{n} y_i \ln\left(\frac{\exp}{1+\exp}\right) + (1-y_i)\ln\left(\frac{1}{1+\exp}\right)$$

$$\cancel{\frac{\partial l}{\partial \beta_1} = \sum_{i=1}^{n} y_i} \qquad \cancel{= \sum_{i=1}^{n} \frac{\partial}{\partial \beta_1} \frac{\exp(\beta_1+\beta_2 x_i)}{1+\exp(\beta_1+\beta_2 x_i)}} =$$

$$= \sum_{i=1}^{n} y_i\left[\ln(\exp) - \ln(1+\exp)\right] + (1-y_i)\left[\ln(1) - \ln(1+\exp)\right]$$

(b)

$$\frac{\partial l}{\partial \beta_1} = \sum_{i=1}^{n} y_i\left[\frac{\exp(\cdot)}{\exp(\cdot)} - \frac{\exp(\cdot)}{1+\exp(\cdot)}\right] + (1-y_i)\left[-\frac{\exp(\cdot)}{1+\exp(\cdot)}\right]$$

$$= \sum_{i=1}^{n} y_i \cancel{- y_i \frac{\exp}{1+\exp}} + -\frac{\exp}{1+\exp} \cancel{+ y_i \frac{\exp}{1+\exp}}.$$

$$\boxed{= \sum_{i=1}^{n} y_i - \Lambda_i(\beta)}$$

$$l = \sum_{i=1}^{n} y_i\left[\ln(\exp(\cdot)) - \ln(1+\exp)\right] + (1-y_i)\left[\ln(1) - \ln(1+\exp(\cdot))\right]$$

$$\frac{\partial l}{\partial \beta_2} = \sum_{i=1}^{n} y_i\left[\frac{x_i \exp(\cdot)}{\exp(\cdot)} - \frac{x_i \exp(\cdot)}{1+\exp(\cdot)}\right] + (1-y_i)\left[-\frac{x_i \exp(\cdot)}{1+\exp(\cdot)}\right]$$

$$= \sum_{i=1}^{n} y_i x_i - \cancel{y_i x_i \Lambda_i(\beta)} - x_i \Lambda_i(\beta) + \cancel{y_i x_i \Lambda_i(\beta)}$$

$$\boxed{= \sum_{i=1}^{n} (y_i - \Lambda_i(\beta)) x_i}$$

(c)

$$\frac{\partial}{\partial \beta_1} \Lambda_i(\beta) = \frac{\partial}{\partial \beta_1} \left[\exp(\beta_1+\beta_2 x_i)\right]\left[1+\exp(\beta_1+\beta_2 x_i)\right]^{-1}$$

$$= -1 \exp(\cdot)\left[1+\exp(\cdot)\right]^{-2} \exp(\cdot) + \exp(\cdot)\left[1+\exp(\beta_1+\beta_2 x_i)\right]^{-1}$$

$$= \frac{(1+\exp)\exp(\cdot) - \exp(\cdot)^2}{(1+\exp(\cdot))^2} = \frac{\exp(\cdot) + \cancel{(\exp(\cdot))^2} - \cancel{\exp(\cdot)^2}}{(1+\exp(\cdot))^2}$$

$$\frac{\partial}{\partial \beta_2} \Lambda_i = -1\, x_i \exp(\cdot)\left[1+\exp(\cdot)\right]^{-2} \exp(\cdot) + x_i \exp(\cdot)\left[1+\exp(\cdot)\right]^{-1}$$

$$= \frac{-x_i \exp(\cdot)^2 + x_i \exp(\cdot)(1+\exp(\cdot))}{(1+\exp(\cdot))^2}$$

$$= x_i \frac{\exp}{(1+\exp)^2}$$

$$\frac{\partial^2 l}{\partial \beta_1^2} = \frac{\partial}{\partial \beta_1}\left[\sum_{i=1}^{n} y_i - \Lambda_i(\beta)\right]$$

$$= \sum_{i=1}^{n} - \frac{\exp(\cdot)}{(1+\exp(\cdot))^2}$$

$$\frac{\partial^2 l}{\partial \beta_2^2} = \frac{\partial}{\partial \beta_2}\left[\sum_{i}^{n} y_i x_i - x_i \Lambda_i(\beta)\right]$$

$$= \sum_{i}^{n} -x_i^2 \frac{\exp}{(1+\exp)^2}$$

$$\frac{\partial^2 l}{\partial \beta_1 \beta_2} = \frac{\partial}{\partial \beta}\left[\sum_{i=1}^{n} y_i - \Lambda_i(\beta)\right]$$

$$= -\sum_{i=1}^{n} x_i \frac{\exp}{(1+\exp)^2}$$

$$H = \begin{bmatrix} -\sum_{i=1}^{n} \frac{\exp}{(1+\exp)^2} & -\sum_{i=1}^{n} x_i \frac{\exp}{(1+\exp)^2} \\ -\sum_{i=1}^{n} x_i \frac{\exp}{(1+\exp)^2} & -\sum_{i=1}^{n} x_i^2 \frac{\exp}{(1+\exp)^2} \end{bmatrix}$$

(d) foc:

① $0 = \sum_{i=1}^{n} (y_i - \Lambda_i(\hat{\beta}))x_i$

② $0 = \sum_{i=1}^{n} y_i - \Lambda_i(\hat{\beta})$

$$\frac{\exp(\hat{\beta}_1 + \hat{\beta}_2 x_i)}{1 + \exp(\hat{\beta}_1 + \hat{\beta}_2 x_i)}$$

~~let x_i~~ $= 0$ $\quad y_i \begin{cases} 0 \\ 1 \end{cases} \quad x_i \begin{cases} 0 \\ 1 \end{cases}$

②

$$\sum_i^n y_i = \sum_{i=1}^{n} \Lambda_i(\hat{\beta})$$

$$= \sum_{\substack{i=1 \\ i|x=0}}^{n} \frac{\exp(\hat{\beta}_0)}{1+\exp(\hat{\beta}_0)}$$

~~if $x=$~~

$$+ \sum_{j:x=1}^{n} \frac{\exp(\hat{\beta}_0 + \hat{\beta}_1)}{1+\exp(\hat{\beta}_0 + \hat{\beta}_1)}$$

①

$$\sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} \Lambda_i(\hat{\beta})\, x_i$$

$$= \underbrace{\sum 0}_{\text{if } x_i=0} + \underbrace{\sum_{i=1} \frac{\exp(\hat{\beta}_0 + \hat{\beta}_1)}{1+\exp(\hat{\beta}_0 + \hat{\beta}_1)}}_{\text{if } x_i=1}$$

suppose $\boxed{\begin{array}{lll} n_1 + n_0 & = & n \\ n_1 & \Rightarrow & x=1 \\ n_0 & \Rightarrow & x=0 \end{array}}$

① $$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \Lambda(\hat{\beta}_i)$$

$$= \sum_{i=1}^{n} \frac{\exp(\hat{\beta}_0 + \hat{\beta}_1 x_i)}{1+\exp(\hat{\beta}_0 + \hat{\beta}_1 x_i)}$$

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n_0} \frac{\exp(\hat{\beta}_0)}{1+\exp(\hat{\beta}_0)} + \sum_{j=1}^{n_1} \frac{\exp(\hat{\beta}_0 + \hat{\beta}_1)}{1+\exp(\hat{\beta}_0 + \hat{\beta}_1)}$$

$$= n_0 \cdot \frac{\exp(\hat{\beta}_0)}{1+\exp(\hat{\beta}_0)} + n_1 \frac{\exp(\hat{\beta}_0 + \hat{\beta}_1)}{1+\exp(\hat{\beta}_0 + \hat{\beta}_1)}$$

② $$\sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} \frac{\exp(\hat{\beta}_0 + \hat{\beta}_1 x_i)}{1+\exp(\hat{\beta}_0 + \hat{\beta}_1 x_i)} x_i$$

$$= \sum_{i=1}^{n_0} \frac{\exp(\hat{\beta}_0)}{1+\exp(\hat{\beta}_0)} \cdot 0 + \sum_{i=1}^{n_1} \frac{\exp(\hat{\beta}_0 + \hat{\beta}_1)}{1+\exp(\hat{\beta}_0 + \hat{\beta}_1)}$$

$$= n_1 \cdot \frac{\exp(\hat{\beta}_0 + \hat{\beta}_1)}{1+\exp(\hat{\beta}_0 + \hat{\beta}_1)}$$

combine ① + ②

$$\sum_{i=1}^{n} y_i = n_0 \cdot \frac{\exp(\hat{\beta}_0)}{1+\exp(\hat{\beta}_0)} + \sum_{i=1}^{n} x_i y_i$$

$$\frac{1}{n_0}\left[\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i y_i\right] = \frac{\exp(\hat{\beta}_0)}{1+\exp(\hat{\beta}_0)}$$

$$\boxed{\hat{\beta}_0 = \ln\left[\frac{\frac{1}{n_0}[\sum y_i - \sum x_i y_i]}{1 - \frac{1}{n_0}[\sum y_i - \sum x_i y_i]}\right]}$$

(5)

(a)

$$y_t = (1 + \theta_1 L + \theta_2 L^2 + \theta_3 L^3) u_t$$

$$\frac{y_t}{1+\theta_1 L+\theta_2 L^2+\theta_3 L^3} = u_t$$

$$\frac{1}{1+\theta_1 L+\theta_2 L^2+\theta_3 L^3} = \phi_0 + \phi_1 L + \phi_2 L^2 + \phi_3 L^3 + \dots$$

$$1 = (1 + \theta_1 L + \theta_2 L^2 + \theta_3 L^3)(\phi_0 + \phi_1 L + \phi_2 L^2 + \phi_3 L^3 + \dots)$$

$$\begin{aligned}
1 &= \phi_0 + \\
0L &= \phi_1 L + \phi_0 \theta_1 L + \\
0L^2 &= \phi_2 L^2 + \phi_1 \theta_1 L^2 + \phi_0 \theta_2 L^2 + \\
0L^3 &= \phi_3 L^3 + \phi_2 \theta_1 L^3 + \phi_1 \theta_2 L^3 + \phi_0 \theta_3 L^3 \\
&\dots
\end{aligned}$$

$$\phi_0 = 1$$
$$\phi_1 = -\phi_0 \theta_1$$
$$\phi_2 = -\phi_1 \theta_1 - \phi_0 \theta_2$$
$$\phi_3 = -\phi_2 \theta_1 - \phi_1 \theta_2 - \phi_0 \theta_3$$

$$\phi_0 = 1$$

$$\phi_1 = -\theta_1$$

$$\phi_2 = \theta_1^2 - \theta_2$$

$$\phi_3 = -\theta_1(\theta_1^2 - \theta_2) - (-\theta_1)\theta_2 - \theta_3 = -\theta_1^3 + 2\theta_1\theta_2 - \theta_3$$

(b)

$$E[y_t] = E[u_t] + \theta_1 E[u_{t-1}] + \theta_2 E[u_{t-2}] + \theta_3 E[u_{t-3}]$$

$$E[y_t] = 0$$

$$Var(y_t) = var(u_t) + \theta_1^2 var(u_{t-1}) + \theta_2^2 var(u_{t-2}) + \theta_3^2 var(u_{t-3})$$

(iid $\therefore$ $cov(u_t, u_s) = 0 \quad \forall\ t \neq s$)

$$var(y_t) = \sigma^2 (1 + \theta_1^2 + \theta_2^2 + \theta_3^2)$$

$$cov(y_t, y_{t-h}) = cov(u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2} + \theta_3 u_{t-3},\ u_{t-h} + \theta_1 u_{t-h-1} + \theta_2 u_{t-h-2} + \theta_3 u_{t-h-3})$$

must have cov up to $h=3$ by construction

$$cov(y_t, y_{t-1}) = cov(u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2} + \theta_3 u_{t-3},\ u_{t-1} + \theta_1 u_{t-2} + \theta_2 u_{t-3} + \theta_3 u_{t-4})$$

$$= \theta_1 var(u_{t-1}) + \theta_1\theta_2 var(u_{t-2}) + \theta_3\theta_2 var(u_{t-3})$$

$$\cancel{= \sigma^2[\theta_1}$$

$$\cancel{= \sigma^2(1 + \theta_1^2 + \theta_2^2 + \theta_3^3)}$$

$$= \sigma^2[\theta_1 + \theta_1\theta_2 + \theta_2\theta_3]$$

$$cov(y_t, y_{t-2}) = cov(u_t + \sum_{i=1}^{3} \theta_i u_{t-i},\ u_{t-2} + \theta_1 u_{t-3} + \theta_2 u_{t-4} + \theta_3 u_{t-5})$$

$$= \theta_2 var(u_{t-2}) + \theta_1\theta_3 var(u_{t-3})$$

$$= \sigma^2[\theta_2 + \theta_1\theta_3]$$

$$cov(y_t, y_{t-3}) = \theta_3 var(u_{t-3}) = \theta_3 \sigma^2$$

$$cov(y_t, y_{t-h}) = 0 \quad \forall h > 3$$

(c)

$$\text{var}(\bar{y}_T) = \frac{1}{T^2}\left[\underbrace{\sum_{t=1}^{T} \text{var}(y_t)}_{①} + \underbrace{2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T} \text{cov}(y_t, y_s)}_{②}\right]$$

$$① = T\sigma^2(1+\theta_1^2+\theta_2^2+\theta_3^2)$$

Intuition:

$$\begin{bmatrix} (1,1) & (1,2) & (1,3) & \cdots & (1,T) \\ (2,1) & (2,2) & (2,3) & \cdots & (2,T) \\ (3,1) & (3,2) & (3,3) & \cdots & (3,T) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (T,1) & (T,2) & (T,3) & \cdots & (T,T) \end{bmatrix}$$

$$= \sum_{t=1}^{T} \text{var}(y_t)$$

$$= \sum_{t=1}^{T-1}\sum_{s=t+1}^{T} \text{cov}(y_t, y_s)$$

needs to be X2 to do whole grid!

$$= \sum_{t=2}^{T} \text{cov}(y_{t-1}, y_t) + \sum_{t=3}^{T} \text{cov}(y_{t-2}, y_t) + \dots + \sum_{t=T}^{T} \text{cov}(y_1, y_t)$$

It works!

$$② = 2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T} \text{cov}(y_t, y_s)$$

~~case A: $s = t-1$~~

$$\cancel{2\sum_{t=1}^{T-1}\sum^{T}}$$

$$2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T} \text{cov}(y_t, y_s) = 2\left[\sum_{t=2}^{T} \text{cov}(y_{t-1}, y_t) + \sum_{t=3}^{T} \text{cov}(y_{t-2}, y_t) + \sum_{t=4}^{T} \text{cov}(y_{t-3}, y_t) + \dots\right]$$

$$= 2\Big[(T-1)\sigma^2[\theta_1 + \theta_1\theta_2 + \theta_2\theta_3] + (T-2)\sigma^2[\theta_2 + \theta_1\theta_3] + (T-3)\sigma^2\theta_3\Big]$$

$$\text{var}(\bar{y}_T) = \frac{1}{T^2}\Big[T\sigma^2(1+\theta_1^2+\theta_2^2+\theta_3^2) + 2(T-1)\sigma^2(\theta_1+\theta_1\theta_2+\theta_2\theta_3) + 2(T-2)\sigma^2[\theta_2+\theta_1\theta_3] + 2(T-3)\sigma^2\theta_3\Big]$$

$$= \sigma^2\Big[\frac{1}{T}(1+\theta_1^2+\theta_2^2+\theta_3^2) + \frac{2(T-1)}{T^2}(\theta_1+\theta_1\theta_2+\theta_2\theta_3) + \frac{2(T-2)}{T^2}[\theta^2+\theta_1\theta_3] + \frac{2(T-3)}{T^2}\theta^3$$

Check consistency by MSC:

$$var(\bar{y}_T) = \frac{1}{T}\sigma^2(1+\theta_1^2+\theta_2^2+\theta_3^2) + \frac{2(T-1)}{T^2}\sigma^2(\theta_1+\theta_2\theta_1+\theta_3\theta_2)$$
$$+ \frac{2(T-2)}{T^2}\sigma^2(\theta_2+\theta_1\theta_3)$$
$$+ \frac{2(T-3)}{T^2}\theta_3\sigma^2$$

$$\frac{1}{T} \xrightarrow{p} 0$$

$$\frac{2(T-1)}{T^2} \longrightarrow \frac{2}{T} \xrightarrow{p} 0$$

$$\frac{2(T-2)}{T^2} \longrightarrow \frac{2}{T} \xrightarrow{p} 0$$

$$\frac{2(T-3)}{T^2} \longrightarrow \frac{2}{T} \xrightarrow{p} 0$$

$$var(\bar{y}_T) \xrightarrow{p} 0$$

$$\mathbb{E}[\bar{y}_T] = \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[y_t] = 0$$

$\therefore$ yes consistent for $\mathbb{E}[y_t]$

(d)

?

Finite: $\bar{y}_T$ is the sum of normally distributed errors hence is normally distributed.

$$\bar{y}_T \sim N(0, var(\bar{y}_T)) = N(0, \tfrac{1}{T}\sigma^2[(1+\theta_1^2+\dots) + \tfrac{2(T-1)}{T}(\theta_1+\dots)\dots])$$

$$\sqrt{T}\,\bar{y}_T = T^{-1/2}\sum_{t=1}^{T} y_t$$

(6)

(a) $y_t = p y_{t-2} + u_t$

$y_{t-2} = p y_{t-4} + u_{t-2}$

$y_{t-4} = p y_{t-6} + u_{t-4}$

$$y_t = p^3 y_{t-6} + u_t + p u_{t-2} + p^2 u_{t-4}$$

$$y_t = p^{n/2} y_{t-n} + u_t + p u_{t-2} + p^2 u_{t-4} + \dots + p^{\frac{n}{2}-1} u_{t-2(n-1)(\frac{n}{2}-1)}$$

t=even $$\boxed{y_t = p^{t/2} y_0 + \sum_{i=0}^{\frac{t}{2}-1} p^i u_{t-2i}}$$ ✓

t= odd? $$\boxed{y_t = p^{\frac{t+1}{2}} y_{-1} + \sum_{i=0}^{\frac{t-1}{2}} p^i u_{t-2i}}$$

Check?

$t=6 \Rightarrow y_6 = p^3 y_0 + u_6 + p u_4 + p^2 u_2$ ✓

$t=7 \Rightarrow y_7 = p^4 y_{0-1} + u_7 + p u_5 + p^2 u_3 + p^3 u_1$ ✓

(b)

$y_t - p y_{t-2} = u_t$

$(1 - pL^2) y_t = u_t$

invertible? $0 = 1 - pL^2$ $\quad L^2 = \frac{1}{p}$ $\quad L = \frac{1}{\sqrt{p}}$

outside unit circle if $|L| > 1$ hence $|\sqrt{p}| < 1$

$\therefore |p| < 1$

hence invertible! ✓

$$\frac{1}{1-\rho L^2} = \left[\phi_0 + \phi_1 L + \phi_2 L^2 + \phi_3 L^3 + \dots\right]$$

$$1 = \phi_0 + \dots$$
$$+ \phi_1 L + \dots$$
$$+ -\phi_0 \rho L^2 + \phi_2 L^2 + \dots$$
$$+ \phi_3 L^3 - \rho \phi_1 L^3 + \dots$$
$$+ \phi_4 L^4 - \phi_2 \rho L^4 + \dots$$

$\phi_0 = 1$

$\phi_1 L = 0L \quad \phi_1 = 0$

$(\phi_2 - \phi_0 \rho) = 0 \Rightarrow (\phi_2 - \rho) = 0 \qquad \phi_2 = \rho$

$(\phi_3 - \phi_1 \rho) = 0 \Rightarrow \phi_3 = 0$

$(\phi_4 - \phi_2 \rho) = 0 \Rightarrow \phi_4 = \rho^2$

$(\phi_6 - \phi_4 \rho) = 0 \qquad \phi_6 = \rho^3$

$$\boxed{y_t = u_t + \rho u_{t-2} + \rho^2 u_{t-4} + \dots}$$

$$\boxed{y_t = \sum_{s=0}^{\infty} \rho^{2s} u_{t-2s}}$$

(c)

$$\mathbb{E}[y_t] = \mathbb{E}[u_t] + \rho\, \mathbb{E}[u_{t-2}] + \rho^2 \mathbb{E}[u_{t-4}] + \dots$$
$$= 0$$

$$var(y_t) = var(u_t) + \rho^2 var(u_{t-2}) + \rho^4 var(u_{t-4}) + \rho^6 var(u_{t-6}) + \dots$$
$$= \sum_{i=0}^{\infty} \sigma_u^2 \rho^{2i}$$

$cov(y_t, y_{t-h}) =$

(7)

(a)

$$\underset{\delta}{\operatorname{argmin}} \sum_{t=1}^{T} (z_t - \delta t)^2$$

foc:

$$\sum_{t=1}^{T} -\cancel{2}t (z_t - \hat{\delta} t) = 0$$

$$\sum_{t=1}^{T} t \cdot z_t = \hat{\delta} \sum_{t=1}^{T} t^2$$

$$\hat{\delta} = \frac{\sum t \cdot z_t}{\sum t^2}$$

$$\cancel{\hat{\delta} = \frac{\sum t (z_{t-1} + \mu + e_t)}{\sum t^2}}$$

$$=$$

$$z_t = \mu + z_{t-1} + e_t$$

$$z_{t-1} = z_{t-2} + \mu + e_{t-1}$$

$$z_{t-2} = \mu + z_{t-3} + e_{t-2}$$

$$z_t = 3\mu + z_{t-3} + e_t + e_{t-1} + e_{t-2}$$

$$z_t = h\mu + z_{t-h} + \sum_{i=0}^{h-1} e_{t-i}$$

$$z_t = t \cdot \mu + \underbrace{z_0}_{=0} + \sum_{i=0}^{t-1} e_{t-i}$$

$$= \sum_{j=1}^{t} e_j$$

$$\hat{\delta} = \frac{\sum_{t=1}^{T} t \cdot (t \cdot \mu + \sum_{j=1}^{t} e_j)}{\sum_{t=1}^{T} t^2} = \mu + \frac{\sum_{t=1}^{T} t \sum_{j=1}^{t} e_j}{\sum_{t=1}^{T} t^2} = \mu + \frac{\sum_{t=1}^{T} v_t}{\sum_{t=1}^{T} t^2}$$

(b)

$$\mathbb{E}[V_t] = \mathbb{E}\left[\left(\sum_{j=1}^{t} e_j\right)t\right] = t\sum_{j=1}^{t}\underbrace{\mathbb{E}[e_j]}_{=0} = 0$$

$$Var(V_t) = Var\left(t\sum_{j=1}^{t} e_j\right) = t^2 \, Var\left(\sum_{j=1}^{t} e_j\right) = t^2 \cdot t \cdot \sigma^2 = t^3\sigma^2$$

$$cov(V_t, V_s) = cov\left(t\sum_{j=1}^{t} e_j \, , \, s\sum_{j=1}^{s} e_j\right) = t \cdot s \cdot cov\left(\sum_{j=1}^{t} e_j \, , \, \sum_{j=1}^{s} e_j\right)$$

$$= cov(e_1 + e_2 + \dots + e_t \, , \, e_1 + e_2 + \dots + e_s)$$

(cov. are 0 for all $i \neq j$ by iid hence only worry about variances)

$$= \min\{t, s\}\,\sigma^2$$

$$\boxed{cov(V_t, V_s) = t \cdot s \cdot \min\{t, s\}\,\sigma^2}$$

(c)

$$\mathbb{E}[\hat{\delta}_T] = \mathbb{E}\left[\mu + \frac{\sum_{t=1}^{T}\left(\sum_{j=1}^{t} e_j\right)t}{\sum_{t=1}^{T} t^2}\right] = \mu + \frac{\sum_{t=1}^{T}\mathbb{E}[V_t]}{\sum_{t=1}^{T} t^2} = \mu$$

$$var(\hat{\delta}_T) = var\left[\frac{\sum_{t=1}^{T} V_t}{\sum_{t=1}^{T} t^2}\right] = \frac{1}{\left[\sum_{t=1}^{T} t^2\right]^2} var\left[\sum_{t=1}^{T} V_t\right]$$

$$var\left[\sum_{t=1}^{T} V_t\right] = \sum_{t=1}^{T} Var(V_t) + 2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T} cov(V_t, V_s)$$

$$= \sum_{t=1}^{T} t^3\sigma^2 + 2\sum_{t=1}^{T-1}\sum_{s=t+1}^{T} t \cdot s \cdot \min\{t, s\}\,\sigma^2$$

$s = t+1 \quad \therefore \min\{t, s\} = t$

$$\operatorname{var}(\hat{\delta}_t) = \frac{1}{\left[\sum_{t=1}^{T} t^2\right]^2}\left[\sum_{t=1}^{T} t^3\sigma^2 + 2\sigma \sum_{t=1}^{T-1} t^2 \sum_{s=t+1}^{T} s\right]$$

$$\sum_{s=t+1}^{T} s = \sum_{s=1}^{T} s - \sum_{s=1}^{t} s$$

$$= \frac{T(T+1)}{2} - \frac{t(t+1)}{2}$$

$$= \frac{1}{\left[\sum_{t=1}^{T} t^2\right]^2}\left[\sum_{t=1}^{T} t^3\sigma^2 + \cancel{2}\sigma \sum_{t=1}^{T-1} t^2\left[\frac{T(T+1)}{\cancel{2}} - \frac{t(t+1)}{\cancel{2}}\right]\right]$$

$$= \frac{\sigma^2}{\left[\sum_{t=1}^{T} t^2\right]^2}\left[\sum_{t=1}^{T} t^3 + \sum_{t=1}^{T} t^2\left[T(T+1) - t(t+1)\right]\right]$$

because when $t = T$

$T^2\left[T(T+1) - T(T+1)\right]$ $\underset{=0}{}$

$= 0$

∴ makes no difference

$\cancel{(T-1)^2\left[T(T+1) - (T-1)(T-1+1)\right]}$

$\cancel{(T-1)^2\left[T(T+1) - T(T-1)\right]}$

$\cancel{T(T-1)^2\left[T+1 - T+1\right]}$

$\cancel{= T(T-1)^2 2}$

(d)

$$var(\hat{\delta}_T) = \frac{\sigma^2}{\left(\sum_{t=1}^{T} t^2\right)^2}\left[\cancel{\sum t^3} + T^2\sum t^2 + T\sum t^2 - \sum t^4 - \cancel{\sum t^3}\right]$$

$$= \frac{\sigma^2}{\left(\sum_{t=1}^{T} t^2\right)^2}\left[(T^2+T)\sum_{t=1}^{T} t^2 - \sum_{t=1}^{} t^4\right]$$

$$= \frac{\sigma^2}{T\left(\frac{1}{T^3}\sum_{t=1}^{T} t^2\right)^2}\left[\frac{T^2+T}{T^5}\sum_{t=1}^{T} t^2 - \frac{1}{T^5}\sum_{t=1}^{T} t^4\right]$$

need this since $\dfrac{\sigma^2}{\left(\frac{1}{T^3}\right)^2} = T^6\sigma^2$

$$\frac{1}{T^3}\sum_{t=1}^{T} t^2 \longrightarrow \int_0^1 r^2\,dr = \left[\tfrac{1}{3}r^3\right]_0^1 = \tfrac{1}{3}$$

$$\frac{T^2+T}{T^5} \approx \frac{1}{T^3} \quad \therefore \quad \frac{T^2+T}{T^5}\sum_{t=1}^{T} t^2 \approx \frac{1}{T^3}\sum_{t=1}^{T} t^2 \rightarrow \tfrac{1}{3}$$

$$\frac{1}{T^5}\sum_{t=1}^{T} t^4 \rightarrow \int_0^1 r^4\,dr = \left[\tfrac{1}{5}r^5\right]_0^1 = \tfrac{1}{5}$$

hence

$$var(\hat{\delta}_T) \approx \frac{\sigma^2}{T\left(\frac{1}{3}\right)^2}\left[\tfrac{1}{3} - \tfrac{1}{5}\right] = \frac{9\sigma^2}{T}\frac{2}{15} = \frac{6}{5}\frac{\sigma^2}{T}$$

$$var(\sqrt{T}\,\hat{\delta}_T) \approx \cancel{T}\,\frac{6}{5}\frac{\sigma^2}{\cancel{T}} = \frac{6\sigma^2}{5}$$

(e)

$$\mathbb{E}[\hat{\sigma}_T] = \mu$$

$$\text{var}[\hat{\sigma}_T] \approx \frac{6}{5}\frac{\sigma^2}{T} \longrightarrow 0$$

$\therefore$ by ~~MSQ~~ MSC $\hat{\sigma}_T$ is consistent for μ!!

(8)

$$y_t = \gamma_1 y_{t-1} + \gamma_2 y_{t-2} + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + u_t$$

(a)

$$(1 - \gamma_1 L - \gamma_2 L^2)\, y_t = (\beta_0 + \beta_1 L + \beta_2 L^2)\, x_t \quad + u_t .$$

invertible?

$$1 - (-0.2)L - 0.4L^2 = 0$$

$L = 1.85$ and -1.35 ∴ outside unit circle

∴ invertible

$$y_t = \frac{0.3 + 0.8L + 0.6L^2}{1 + 0.2L - 0.4L^2}\, x_t + \frac{u_t}{1 + 0.2L - 0.4L^2}$$

- Impact multiplier:

$$m_0 = \frac{\Delta y_t}{\Delta x_t} = \frac{B_r(0)}{C_p(0)} = \frac{0.3}{1} = \boxed{0.3}$$

- Total multiplier:

$$m_{total} = \frac{B_r(1)}{C_p(1)} = \frac{0.3 + 0.8 + 0.6}{1 + 0.2 - 0.4} = \frac{1.7}{0.8} = \boxed{2.125}$$

- Mean lag:

$$\frac{\left.\frac{dB(L)}{dL}\right|_{L=1}}{B(1)} - \frac{\left.\frac{dC(L)}{dL}\right|_{L=1}}{C(1)} = \frac{0.8 + 2\cdot 0.6}{0.3 + 0.8 + 0.6} - \frac{0.2 - 2\cdot 0.4}{1 + 0.2 - 0.4}$$

$$\boxed{= 2.92}$$

Median lag:

$$\frac{0.3 + 0.8L + 0.6L^2}{1 + 0.2L - 0.4L^2} = \delta^0 + \delta^1 L + \delta^2 L^2 + \delta^3 L^3 + \dots$$

$$0.3 + 0.8L + 0.6L^2 = (1 + 0.2L - 0.4L^2)(\delta_0 + \delta_1 L + \delta_2 L^2 + \delta_3 L^3 + \dots)$$

$$0.3 = \delta_0$$

$$0.8L = \delta_1 L + 0.2\delta_0 L$$

$$\delta_1 = 0.8 - 0.2 \cdot 0.3 = 0.74$$

$$\cancel{0.6L^2 = \delta_0 0.4L^2 + 0.2\delta_1 L^2}$$

$$0.6L^2 = \delta_2 L^2 + \delta_1 0.2 L^2 - \delta_0 0.4L^2$$

$$\delta_2 = 0.6 - (0.74 \cdot 0.2) + 0.3 \cdot 0.4$$

$$\delta_2 = 0.868$$

$$0L^3 = \delta_3 L^3 + 0.2\delta_2 L^3 - 0.4\delta_1 L^3$$

$$\delta_3 = 0 - 0.2(0.868) + 0.4(0.74)$$

$$\delta_3 = 0.1224$$

$$0.3 + 0.74L + 0.868L^2 + 0.1224L^3 + \dots$$

$$\min_q \frac{\sum_{i=0}^{q} \delta_i}{2.125} \geq 0.5 \qquad \min_q \sum_{i=0}^{q} \delta_i \geq \frac{17}{16}$$

$$0.3 + 0.74 < \frac{17}{16}$$

$$0.3 + 0.74 + 0.868 > \frac{17}{16}$$

$$\boxed{\therefore q = 2}$$

(b)

ECM:

$$y_t - y_{t-1} = (\gamma_1 - 1) y_{t-1} \cancel{+} \gamma_2 (y_{t-1} - y_{t-2}) + \gamma_2 y_{t-1} + \beta_0 (x_t - x_{t-1}) + \beta_0 x_{t-1}$$
$$+ \beta_1 x_{t-1} \cancel{+} - \beta_2 (x_{t-1} - x_{t-2})$$
$$+ \beta_2 x_{t-1} + u_t$$

$$\Delta y_t = (\gamma_2 + \gamma_1 - 1) y_{t-1} - \gamma_2 \Delta y_{t-1} + \beta_0 \Delta x_t + (\beta_0 + \beta_1 + \beta_2) x_{t-1}$$
$$- \beta_2 \Delta x_{t-1} + u_t$$

$$\Delta y_t = (\gamma_2 + \gamma_1 - 1) \left[y_{t-1} + \frac{\beta_0 + \beta_1 + \beta_2}{(\gamma_2 + \gamma_1 - 1)} x_{t-1} \right] + \beta_0 \Delta x_t - \gamma_2 \Delta y_{t-1} - \beta_2 \Delta x_{t-1} + u_t.$$

(c)

$$\mathbb{E}[x_t] = \mu$$

Stationary implies $\mathbb{E}\, \Delta y_t = 0$ $\quad \mathbb{E}[\Delta x_t] = 0$ $\quad \mathbb{E}[\Delta y_{t-1}] = 0$ $\quad \mathbb{E}[\Delta x_{t-1}] = 0$

$$\Rightarrow \mathbb{E}[y_{t-1}] = -\frac{\beta_0 + \beta_1 + \beta_2}{\gamma_2 + \gamma_1 - 1} \mu$$

$$\boxed{\mathbb{E}[y_t] = \frac{\beta_0 + \beta_1 + \beta_2}{1 - \gamma_1 - \gamma_2} \mu}$$

(d)

$$\gamma_2 = \beta_1 = \beta_2 = 0$$

$$\Delta y_t = (\gamma_1 - 1) \left[y_{t-1} \cancel{+} - \left(\frac{\beta_0}{1 - \gamma_1} x_{t-1} \right) \right] + \beta_0 \Delta x_t + u_t$$

let $\theta = -\dfrac{\beta_0}{1 - \gamma_1}$

$$\Delta y_t = \overset{(\gamma_1-1)}{\cancel{(1-\gamma_1)}}\left[y_{t-1} - \theta x_{t-1}\right] + \beta_0 \Delta x_t + u_t$$

$$\Delta y_t - \theta \Delta x_t = \overset{(\gamma_1-1)}{\cancel{(1-\gamma_1)}}\left[y_{t-1} - \theta x_{t-1}\right] + (\beta_0 - \theta)\Delta x_t + u_t.$$

let $v_t = y_t - \theta x_t$

$$\cancel{\Delta v_t = \overset{(\gamma_1-1)}{(1-\gamma_1)} v_{t-1} + \underbrace{(\beta_0-\theta)\Delta x_t}_{} + u_t}$$

$$y_t - y_{t-1} - (\theta x_t - \theta x_{t-1}) = (\gamma_1 - 1).\, v_{t-1} + (\beta_0 - \theta)\Delta x_t + u_t.$$

$$y_t - \theta x_t - (y_{t-1} - \theta x_{t-1}) = \gamma_1 v_{t-1} - v_{t-1} + (\beta_0 - \theta)\Delta x_t + u_t.$$

$$v_t - \cancel{v_{t-1}} = \gamma_1 v_t - \cancel{v_{t-1}} + (\beta_0 - \theta)\Delta x_t + u_t.$$

$$v_t = \gamma_1 v_t + \underbrace{(\beta_0 - \theta)\Delta x_t}_{I(0)} + u_t$$

AR process is stationary

hence

$$v_t \sim I(0)$$

$x_t \sim I(1)$

y_t ?

$$\Delta y_t = (\gamma_1 - 1)\underbrace{(y_{t-1} - \theta x_{t-1})}_{I(0)} + \beta_1 \underbrace{\Delta x_t}_{I(0)} + u_t.$$

$\therefore \Delta y_t \sim I(0)$ $\therefore$ $y_t \sim I(1)$

$\therefore$ cointegrated.

Econometrics 2020

1.

(a) $\hat{\varepsilon}_i = y_i - \hat{\beta}$ $\qquad$ $\hat{\beta} = \arg\min_{\beta} \sum (y_i - \beta)^2$

$-2\sum(y_i - \hat{\beta}) = 0$

$\sum y_i = \sum \hat{\beta}$

$\boxed{\hat{\beta} = \frac{1}{n}\sum_{i=1}^{n} y_i}$

$\hat{\varepsilon}_i = y_i - \frac{1}{n}\sum_{i=1}^{n} y_i$

$y_i = \beta + \varepsilon_i$

$\hat{\varepsilon}_i = \beta + \varepsilon_i - \frac{1}{n}\sum \beta + \varepsilon_i$

$= \beta - \frac{1}{n} n\beta + \varepsilon_i - \frac{1}{n}\sum \varepsilon_i$

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(\varepsilon_i - \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\right)^2$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}\left(\varepsilon_i^2 + \frac{1}{n^2}\left[\sum_{i=1}^{n}\varepsilon_i\right]^2 - 2\frac{1}{n}\varepsilon_i\sum_{i=1}^{n}\varepsilon_i\right)$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}\left(\varepsilon_i^2 + \bar{\varepsilon}^2 - 2\varepsilon_i\bar{\varepsilon}\right)$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n}\varepsilon_i^2 + n\bar{\varepsilon}^2 - 2\bar{\varepsilon}\sum_{i=1}^{n}\varepsilon_i\right] \qquad = -2\bar{\varepsilon}\cdot n\bar{\varepsilon} = -2n\bar{\varepsilon}^2$$

$$= \frac{1}{n-1}\left[\sum \varepsilon_i^2 - n\bar{\varepsilon}^2\right]$$

$$\mathbb{E}[\hat{\sigma}^2] = \frac{1}{n-1}\left[\sum_{i=1}^{n}\underbrace{\mathbb{E}[\varepsilon_i^2]}_{=\sigma_i^2} - n\mathbb{E}[\bar{\varepsilon}^2]\right]$$

$$\mathrm{var}(\bar{\varepsilon}) = \mathbb{E}[\bar{\varepsilon}^2] - (\mathbb{E}[\bar{\varepsilon}])^2$$

$$\mathbb{E}[\bar{\varepsilon}^2] = var(\bar{\varepsilon}) + (\mathbb{E}[\bar{\varepsilon}])^2$$

$$= var\left(\frac{1}{n}\sum \varepsilon_i\right) \qquad = \left(\frac{1}{n}\sum \mathbb{E}[\varepsilon_i]\right)^2$$

$$= \frac{1}{n^2}\cdot n\,\sigma_\varepsilon^2 \qquad = 0$$

$$= \frac{1}{n}\sigma_\varepsilon^2$$

$$\mathbb{E}[\bar{\varepsilon}^2] = \frac{\sigma^2}{n}$$

$$\mathbb{E}[\hat{\sigma}^2] = \frac{1}{n-1}\left(n\sigma_\varepsilon^2 - n\frac{\sigma_\varepsilon^2}{n}\right)$$

$$= \frac{1}{\cancel{n-1}}\sigma_\varepsilon^2\cancel{(n-1)}$$

$$\boxed{\mathbb{E}[\hat{\sigma}^2] = \sigma_\varepsilon^2} \qquad \therefore \text{unbiased.}$$

(b)

$$\hat{\sigma}_\varepsilon^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n}(\varepsilon_i^2) - n\bar{\varepsilon}^2\right] = \frac{n}{n-1}\left[\frac{1}{n}\sum_{i=1}^{n}[\varepsilon_i^2] - \bar{\varepsilon}^2\right]$$

$$\frac{n}{n-1} \xrightarrow{p} 1$$

$$\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^2 \xrightarrow{p} \mathbb{E}[\varepsilon_i^2] = \sigma_\varepsilon^2 \quad \text{by iid LLN.}$$

$$\bar{\varepsilon}^2 = \left[\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\right]^2 \xrightarrow{p} (\mathbb{E}[\varepsilon_i])^2 = 0 \quad \text{by iid LLN}$$

$$\therefore \boxed{\hat{\sigma}_\varepsilon^2 \xrightarrow{p} \sigma_\varepsilon^2} \qquad \therefore \text{consistent.}$$

(c)

$H_0: \quad \beta = 0$

$$t = \frac{\hat{\beta} - \beta}{s.e.(\hat{\beta})} = \frac{\hat{\beta} - 0}{s.e.(\hat{\beta})}$$

$$var(\hat{\beta}) = var\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) = \frac{1}{n^2} var\left(\sum_{i=1}^{n} \beta + \varepsilon_i\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} var(\varepsilon_i) = \frac{1}{n^2} n \cdot \sigma_\varepsilon^2 = \frac{\sigma_\varepsilon^2}{n}$$

$$s.d.(\hat{\beta}) = \sqrt{\sigma_\varepsilon^2 / n}$$

$$\boxed{s.e.(\hat{\beta}) = \sqrt{\hat{\sigma}_\varepsilon^2 / n}}$$

$$\hat{\beta} - \beta = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i \qquad \sqrt{n}(\hat{\beta} - \beta) = \frac{\sqrt{n}}{n}\sum_{i=1}^{n}(\varepsilon_i - 0)$$

$$\sqrt{n} \quad n^{-1/2}\sum_{i=1}^{n} \varepsilon_i \xrightarrow{D} N(0, \sigma_\varepsilon^2)$$

iid CLT

$$\therefore \sqrt{n}(\hat{\beta} - \beta) \xrightarrow{D} N(0, \sigma_\varepsilon^2)$$

$\therefore$

$$t = \frac{\hat{\beta}}{s.e.(\hat{\beta})} = \frac{(\hat{\beta} - 0)}{\hat{\sigma}_\varepsilon^2 / \sqrt{n}} = \frac{\sqrt{n}(\hat{\beta} - 0)}{\hat{\sigma}_\varepsilon}$$

$$\hat{\sigma}_\varepsilon \xrightarrow{P} \sigma_\varepsilon$$

$$\sqrt{n}(\hat{\beta} - 0) \xrightarrow{D} N(0, \sigma_\varepsilon^2$$

$$\therefore t \xrightarrow{D} \frac{N(0, \sigma_\varepsilon^2)}{\sigma_\varepsilon} = N(0, 1)$$

calc. $t = \dfrac{\hat{\beta} - 0}{s.e.(\hat{\beta})} \overset{a}{\sim} N(0,1)$ under H_0.

reject H_0 if $|t| > CV_\alpha$ where α = sig. level
(two tailed test)

$\alpha = 0.05$ $\quad CV_{0.05} = 1.96$

(d)

$$\hat{\sigma}^2_\varepsilon = \frac{1}{n-1}\left[\sum_{i=1}^{n} \varepsilon_i^2 - n\bar{\varepsilon}^2\right] = \frac{n}{n-1}\frac{1}{n}\sum_{i=1}^{n} \varepsilon_i^2 - \frac{n}{n-1}\bar{\varepsilon}^2$$

$$\hat{\sigma}^2_\varepsilon - \sigma^2_\varepsilon = \frac{n}{n-1}\frac{1}{n}\sum_{i=1}^{n}\left((\varepsilon_i^2 - \sigma^2_\varepsilon) + \sigma^2_\varepsilon\right) - \frac{n}{n-1}\bar{\varepsilon}^2 - \sigma^2_\varepsilon$$

$$= \frac{n}{n-1}\frac{1}{n}\sum_{i=1}^{n}(\varepsilon_i^2 - \sigma^2_\varepsilon) + \frac{n}{n-1}\sigma^2_\varepsilon - \sigma^2_\varepsilon - \frac{n}{n-1}\bar{\varepsilon}^2$$

$$\sqrt{n}(\hat{\sigma}^2_\varepsilon - \sigma^2_\varepsilon) = \frac{n}{n-1}\underbrace{n^{-1/2}\sum_{i=1}^{n}(\varepsilon_i^2 - \sigma^2_\varepsilon)}_{A} + \underbrace{\sqrt{n}\frac{n}{n-1}\sigma^2_\varepsilon}_{B} - \underbrace{\sqrt{n}\,\sigma^2_\varepsilon}_{C} - \underbrace{\sqrt{n}\frac{n}{n-1}\bar{\varepsilon}^2}_{D}$$

$\frac{n}{n-1} \xrightarrow{p} 1 \qquad \sqrt{n} \xrightarrow{p} 0$

(A) $n^{-1/2}\sum_{i=1}^{n}(\varepsilon_i^2 - \sigma^2_\varepsilon) \xrightarrow{D} N(0, var(\varepsilon_i^2))$

(B) $\sqrt{n}\frac{n}{n-1}\sigma^2_\varepsilon \xrightarrow{p} 0$

(C) $\sqrt{n}\,\sigma^2_\varepsilon \xrightarrow{p} 0$

(D) $\bar{\varepsilon}^2 \xrightarrow{p} 0$, $\sqrt{n} \xrightarrow{p} 0$ $\quad \frac{n}{n-1} \xrightarrow{p} 1$ $\quad \therefore \sqrt{n}\frac{n}{n-1}\bar{\varepsilon}^2 \xrightarrow{p} 0$

$\therefore$

$$\sqrt{n}(\hat{\sigma}^2_\varepsilon - \sigma^2_\varepsilon) \xrightarrow{D} N(0, var(\varepsilon_i^2))$$

$$\operatorname{var}(\varepsilon_i^2) = \mathbb{E}[\varepsilon_i^4] - (\mathbb{E}[\varepsilon_i^2])^2$$

$$= (\mu_4 - \sigma_\varepsilon^4)$$

$$\boxed{\sqrt{n}\,(\hat{\sigma}_\varepsilon^2 - \sigma_\varepsilon^2) \xrightarrow{D} N(0, \mu_4 - \sigma_\varepsilon^4)}$$

(2)

$$Y = X\beta + u$$

(a)

$$\hat{\beta} = (Z'X)^{-1} Z'Y$$

$$= (Z'X)^{-1} Z'X\beta + (Z'X)^{-1} Z'u$$

$$= \beta + (Z'X)^{-1} Z'u$$

$$\mathbb{E}[\hat{\beta} \mid X] = \beta + \mathbb{E}[(Z'X)^{-1} Z'u \mid X]$$

Z is a function of X here

$$= \beta + (Z'X)^{-1} Z' \underbrace{\mathbb{E}[u \mid X]}_{=0}$$

$$\boxed{\mathbb{E}[\hat{\beta} \mid X] = \beta}$$

(b)

$$\mathrm{var}(\hat{\beta} \mid X) = \underbrace{\mathrm{var}(\beta \mid X)}_{=0} + \mathrm{var}((Z'X)^{-1} Z'u \mid X)$$

$$= (Z'X)^{-1} Z' \, \mathrm{var}(u \mid X) \left[(Z'X)^{-1} Z'\right]'$$

$$= (Z'X)^{-1} Z' \, \sigma_u^2 \, Z (X'Z)^{-1}$$

$$\boxed{\mathrm{Var}(\hat{\beta} \mid X) = \sigma_u^2 \, (Z'X)^{-1} Z'Z (X'Z)^{-1}}$$

(c)

$$\text{Var}(\hat{\beta}|X) - \text{Var}(\beta|X) = \sigma_u^2 (Z'X)^{-1} Z'Z(X'Z)^{-1} - \sigma_u^2 (X'X)^{-1}$$

$$= \sigma_u^2 \left[(Z'X)^{-1} Z'Z(X'Z)^{-1} - (X'X)^{-1} \right]$$

let $A' = (Z'X)^{-1} Z'$

$$= \sigma_u^2 \left[A'A - A'X(X'X)^{-1}X'A \right]$$

$$= \sigma_u^2 A' \underbrace{\left[I - X(X'X)^{-1}X' \right]}_{=\text{ annihilator matrix } = M} A$$

$$= \sigma_u^2 A'MA$$

Quadratic form

and for any A $A'MA$ is positive semi-definite.

$\Rightarrow$ $\sigma^2 A'MA$ is positive

$\Rightarrow$ $\text{Var}(\hat{\beta}|X) \geq \text{Var}(\beta|X)$

Also know this by Gauss-markov...

(d)

$$F = \frac{(R\hat{\beta} - q)\left\{R\,\hat{\sigma}_u^2\,(X'X)^{-1}R'\right\}^{-1}(R\hat{\beta} - q)}{J} \sim \cancel{F_{n-k-1,\infty}}$$

$$\underline{\underline{F_{J,\,n-k-1}}}$$

(3)

(a)

$$\mathbb{E}[\bar{u}_i] = \frac{1}{m_i}\sum_{e=1}^{m_i} \mathbb{E}[u_{i,e}] = \boxed{0}$$

$$var(\bar{u}_i) = \frac{1}{m_i^2}\sum_{e=1}^{m_i} var(u_{i,e}) = \frac{1}{m_i^2} m_i \sigma_u^2 = \boxed{\frac{\sigma_u^2}{m_i}}$$

$$cov(\bar{u}_i, \bar{u}_j) = \frac{1}{m_i m_j}\sum_{e=1}^{m_i}\sum_{e=1}^{m_j} \overset{cov}{\cancel{var}}(u_{i,e}, u_{j,e}) = \boxed{0} \quad \forall i \neq j$$

(b)

$$\beta \quad \text{argmin} \sum_{i=1}^{n} (\bar{y}_i - \beta_0 - \beta_1 \bar{x}_i)^2$$

foc:

$$-2\sum_{i=1}^{n}(\bar{y}_i - \hat{\beta}_0 - \hat{\beta}_1\bar{x}_i) = 0 \qquad \hat{\beta}_0 = \frac{1}{n}\sum_{i=1}^{n}\bar{y}_i + \hat{\beta}_1\frac{1}{n}\sum_{i=1}^{n}\bar{x}_i$$

$$-2\sum_{i=1}^{n} \cancel{\hat{\beta}_1}\,\bar{x}_i(\bar{y}_i - \hat{\beta}_0 - \hat{\beta}_1\bar{x}_i) = 0$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(\bar{y}_i - \bar{\bar{y}})(\bar{x}_i - \bar{\bar{x}})}{\sum_{i=1}^{n}(\bar{x}_i - \bar{\bar{x}})^2} \qquad \bar{\bar{x}} = \frac{1}{n}\sum_{i=1}^{n}\bar{x}_i$$

$$\hat{\beta}_1 = \beta_1 + \frac{\sum_{i=1}^{n}(\bar{x}_i - \bar{\bar{x}})(\bar{u}_i - \bar{\bar{u}})}{\sum_{i=1}^{n}(\bar{x}_i - \bar{\bar{x}})^2}$$

$$\mathbb{E}[\hat{\beta}] = \beta_1 + \frac{\sum_{i=1}^{n}(\bar{x}_i - \bar{\bar{x}})\,\mathbb{E}[\bar{u}_i]}{\sum(\bar{x}_i - \bar{\bar{x}})^2} = \boxed{\beta_1}$$

$$var(\hat{\beta}) = \frac{1}{\left[\sum_{i=1}^{n}(\bar{x}_i - \bar{\bar{x}})^2\right]^2} var\left(\sum(\bar{x}_i - \bar{\bar{x}})\bar{u}_i\right)$$

$x_{i,e}$ is non-random hence passes through var?

$$var(\hat{\beta}_1) = \frac{1}{\left[\sum(\bar{x}_i - \bar{\bar{x}})^2\right]^2} \sum_{i=1}^{n} (\bar{x}_i - \bar{\bar{x}})^2 var(\bar{u}_i)$$

$$var(\bar{u}_i) = \sigma_u^2 / m_i$$

$$\boxed{var(\hat{\beta}_1) = \frac{\sigma_u^2 \sum_{i=1}^{n} \frac{(\bar{x}_i - \bar{\bar{x}})^2}{m_i}}{\left[\sum_{i=1}^{n} (\bar{x}_i - \bar{\bar{x}})^2\right]^2}}$$

(c)

$$\mathbb{E}[\bar{u}_i^*] = \mathbb{E}\left[\frac{\bar{u}_i}{\sqrt{h_i}}\right] = \boxed{0}$$

$$var(\bar{u}_i^*] = \frac{1}{h_i} var(\bar{u}_i) = \frac{1}{h_i} \frac{\sigma_u^2}{m_i} = \frac{m_i}{1} \frac{\sigma_u^2}{m_i} = \boxed{\sigma_u^2}$$

$$cov(\bar{u}_i^*, \bar{u}_j^*) = \frac{1}{h_i} \frac{1}{h_j} cov(\bar{u}_i, \bar{u}_j) = \boxed{0}$$

(d)

$$(\breve{\beta}_0, \breve{\beta}_1) = \underset{(\beta_0, \beta_1)}{\arg\min} \left[\sum_{i=1}^{n} \left(\frac{\bar{y}_i}{\sqrt{h_i}} - \beta_0 \frac{1}{\sqrt{h_i}} - \beta_1 \frac{\bar{x}_i}{\sqrt{h_i}}\right)^2\right]$$

foc's:

$$0 = \cancel{-2} \sum_{i=1}^{n} \frac{1}{\sqrt{h_i}} \left(\frac{\bar{y}_i}{\sqrt{h_i}} - \tilde{\beta}_0 \frac{1}{\sqrt{h_i}} - \tilde{\beta}_1 \frac{\bar{x}_i}{\sqrt{h_i}}\right)$$

$$0 = \cancel{-2} \sum_{i=1}^{n} \frac{\bar{x}_i}{\sqrt{h_i}} \left(\frac{\bar{y}_i}{\sqrt{h_i}} - \tilde{\beta}_0 \frac{1}{\sqrt{h_i}} - \tilde{\beta}_1 \frac{\bar{x}_i}{\sqrt{h_i}}\right)$$

$$\sum_{i=1}^{n} \left(\frac{\bar{y}_i}{h_i} - \tilde{\beta}_0 \frac{1}{h_i} - \tilde{\beta}_1 \frac{\bar{x}_i}{h_i} \right) = 0$$

$$\frac{1}{h_i} = m_i$$

$$\boxed{\tilde{\beta}_0 = \frac{\sum_{i=1}^{n} m_i \bar{y}_i - \tilde{\beta}_1 \sum_{i=1}^{n} \bar{x}_i m_i}{\sum_{i=1}^{n} m_i}}$$

$$\sum_{i=1}^{n} m_i \bar{x}_i \bar{y}_i - \hat{\beta}_0 \sum m_i \bar{x}_i - \hat{\beta}_1 \sum m_i \bar{x}_i^2 = 0$$

$$\sum_{i=1}^{n} m_i \bar{x}_i \bar{y}_i - \frac{\sum_{i=1}^{n} m_i \bar{y}_i \sum_{i}^{n} m_i \bar{x}_i}{\sum_{i=1}^{n} m_i} + \tilde{\beta}_1 \frac{\left[\sum_{i=1}^{n} \bar{x}_i m_i\right]^2}{\sum_{i=1}^{n} m_i} - \tilde{\beta}_1 \sum_{i=1}^{n} m_i \bar{x}_i^2 = 0$$

$$\tilde{\beta}_1 \left[\left[\sum_{i=1}^{n} \bar{x}_i m_i\right]^2 - \sum_{i=1}^{n} m_i \sum_{i=1}^{n} m_i \bar{x}_i^2 \right] = \sum_{i=1}^{n} m_i \bar{y}_i \sum_{i=1}^{n} m_i \bar{x}_i - \sum_{i=1}^{n} m_i \sum_{i=1}^{n} m_i \bar{x}_i \bar{y}_i$$

$$\boxed{\tilde{\beta}_1 = \frac{\sum_{i=1}^{n} m_i \bar{y}_i \sum_{i=1}^{n} m_i \bar{x}_i - \sum_{i=1}^{n} m_i \sum_{i=1}^{n} m_i \bar{x}_i \bar{y}_i}{\left[\sum_{i=1}^{n} \bar{x}_i m_i\right]^2 - \sum_{i=1}^{n} m_i \sum_{i=1}^{n} m_i \bar{x}_i^2}}$$

$\hat{\beta}_1$

1

(e)

• Use $\hat{\tilde{\beta}}_1$

$\hat{\beta}_1$ has heteroskedastic error ~~s.e.~~ since

$$\boxed{var(\bar{u}_i) = \frac{\sigma_u^2}{m_i}}$$

which depends on m_i (varies with i)

$\hat{\tilde{\beta}}_1$ has homoskedastic error ~~s.e.~~ since

$$\boxed{var(\bar{u}_i^*) = \sigma_u^2}$$

which does not depend on i

$\Rightarrow$ $\hat{\beta}_1$ has larger variance & ~~to~~ $\therefore$ less efficient

$\therefore$ use $\hat{\tilde{\beta}}_1$!

CHECK SOCs

(4)

(a)

$$f_\theta(y_1, \dots, y_n) = \prod_{i=1}^{n} (1/\theta)\, e^{-\frac{1}{\theta} y_i}$$

$$= (1/\theta)^n \cdot e^{-1/\theta \sum_{i=1}^n y_i}$$

$$L_{y_1,\dots,y_n}(\theta) = (1/\theta)^n\, e^{-1/\theta \sum_{i=1}^n y_i}$$

$$\ln l_{y_1,\dots,y_n}(\theta) = n \ln(1/\theta) + -1/\theta \sum_{i=1}^n y_i$$

$$\cancel{\frac{\partial l_{y_1,\dots,y_n}(\theta)}{\partial\theta}} =$$

$$l_{y_1,\dots,y_n}(\theta) = n\ln(1) - n\ln\theta - 1/\theta \sum_{i=1}^n y_i$$

$$= -n\ln\theta - 1/\theta \sum_{i=1}^n y_i$$

$$\frac{\partial l_{y_1,\dots,y_n}(\theta)}{\partial\theta} = -\frac{n}{\hat\theta} + 1/\hat\theta^2 \sum_{i=1}^n y_i = 0$$

$$\frac{\hat\theta^2}{\hat\theta} = \frac{\sum_{i=1}^n y_i}{n}$$

$$\boxed{\hat\theta = \frac{\sum_{i=1}^n y_i}{n} = \bar y}$$

(b)

$$f_y(\theta) = \binom{k}{y} \theta^{y} (1-\theta)^{k-y} \qquad \binom{k}{y} = \frac{k!}{y!(k-y)!}$$

$\uparrow$ doesn't depend on θ.

$$f_{y_1,\dots,y_n}(\theta) = \prod_{i=1}^{n} \binom{k}{y} \theta^{y_i} (1-\theta)^{k-y_i}$$

$$L_{y_1,\dots,y_n}(\theta) = \binom{k}{y}^n \theta^{\sum_{i=1}^n y_i} (1-\theta)^{\sum_{i=1}^n k-y_i}$$

$$l_{y_1,\dots,y_n}(\theta) = n \ln\binom{k}{y} + \sum_{i=1}^n y_i \ln\theta + \sum_{i=1}^n (k-y_i) \ln(1-\theta)$$

$$\frac{\partial l_{y_1,\dots,y_n}(\theta)}{\partial\theta} = \frac{\sum_{i=1}^n y_i}{\hat\theta} + \sum_{i=1}^n (k-y_i) \frac{-1}{1-\hat\theta} = 0$$

$$(1-\hat\theta) \sum_{i=1}^n y_i = \hat\theta \sum_{i=1}^n (k-y_i)$$

$$\sum_{i=1}^n y_i - \hat\theta \sum_{i=1}^n y_i = \hat\theta \sum_{i=1}^n (k-y_i)$$

$$\hat\theta = \frac{\sum_{i=1}^n y_i}{\sum_{i=1}^n y_i + \sum_{i=1}^n (k-y_i)} = \frac{\sum_{i=1}^n y_i}{\cancel{\sum_{i=1}^n y_i} - \cancel{\sum_{i=1}^n y_i} + nk}$$

$$\boxed{\hat\theta = \frac{\sum_{i=1}^n y_i}{nk} = \frac{\bar y}{k}}$$

(c)

$$\theta = \frac{\exp(\beta)}{1+\exp(\beta)}$$

$$f(y_i, \theta) = \theta^{y}(1-\theta)^{1-y}$$

$$f_{y_1,\dots,y_n}(\theta) = \prod_{i=1}^{n} \theta^{y_i}(1-\theta)^{1-y_i}$$

$$= \theta^{\sum_{i=1}^{n} y_i}(1-\theta)^{\sum_{i=1}^{n}(1-y_i)}$$

$$L_{y_1,\dots,y_n}(\theta) = \theta^{\sum_{i=1}^{n} y_i}(1-\theta)^{\sum_{i=1}^{n}(1-y_i)}$$

$$l_{y_1,\dots,y_n}(\theta) = \sum_{i=1}^{n} y_i \ln\theta + \sum_{i=1}^{n}(1-y_i)\ln(1-\theta)$$

(one method)

$$\left[\frac{\partial l_{y_1,\dots,y_n}(\theta)}{\partial\beta} = \sum_{i=1}^{n} y_i \frac{\partial \ln\theta}{\partial\theta}\frac{\partial\theta}{\partial\beta} + \sum_{i=1}^{n}(1-y_i)\frac{\partial\ln(1-\theta)}{\partial\theta}\frac{\partial\theta}{\partial\beta}\right]$$

$$\frac{\partial\theta}{\partial\beta} = \frac{\partial}{\partial\beta} e^{\beta}\cdot(1+e^{\beta})^{-1}$$

$$= \frac{e^{\beta}}{(1+e^{\beta})^2} = \theta(1-\theta)$$

$$\ln\frac{e^{\beta}}{1+e^{\beta}} = \beta - \ln(1+e^{\beta})$$

$$\ln\left(1-\frac{e^{\beta}}{1+e^{\beta}}\right) = \ln\left(\frac{1+e^{\beta}-e^{\beta}}{1+e^{\beta}}\right) = \ln 1 - \ln(1+e^{\beta})$$

$$l_{y_1,\dots,y_n}(\beta) = \sum_{i=1}^{n} y_i\left[\beta - \ln(1+e^{\beta})\right] + \sum_{i=1}^{n}(1-y_i)\left[-\ln(1+e^{\beta})\right]$$

$$\frac{\partial l_{y_1,\dots,y_n}(\beta)}{\partial \beta} = \sum_{i=1}^{n} y_i \left[1 - \frac{e^{\hat{\beta}}}{1+e^{\hat{\beta}}}\right] + \sum_{i=1}^{n} (1-y_i)\left[\frac{-e^{\hat{\beta}}}{1+e^{\hat{\beta}}}\right]$$

$$0 = \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} y_i \frac{e^{\hat{\beta}}}{1+e^{\hat{\beta}}} - \sum_{i=1}^{n} (1-y_i) \frac{e^{\hat{\beta}}}{1+e^{\hat{\beta}}}$$

$$\hat{\theta} = \frac{e^{\hat{\beta}}}{1+e^{\hat{\beta}}} = \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} y_i + \sum_{i=1}^{n}(1-y_i)} = \frac{\sum_{i=1}^{n} y_i}{n} = \bar{y}$$

$$e^{\hat{\beta}} = \bar{y} + e^{\hat{\beta}} \cdot \bar{y} \Rightarrow e^{\hat{\beta}}(1-\bar{y}) = \bar{y}$$

$$\boxed{\hat{\beta} = \ln\left(\frac{\bar{y}}{1-\bar{y}}\right)}$$

(d)

$$f_{y_i}(\theta) = \frac{1}{\theta} 1_{(0 \leq y_i \leq \theta)}$$

$$f_{y_1,\dots,y_n}(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} 1_{(0 \leq y_i \leq \theta)}$$

$$L_{y_1,\dots,y_n}(\theta) = \frac{1}{\theta^n} \prod_{i=1}^{n} 1_{(0 \leq y_i \leq \theta)}$$

need to maximise this function

$\theta \geq y_i$ for $i = 1, \dots, n$ since otherwise $1_{(0 \leq y_i \leq \theta)} = 0$ and hence function is not maximised.

$$\prod_{i=1}^{n} 1_{(0 \leq y_i \leq \theta)} = 1_{0 \leq y_1 \leq \theta} \times \dots \times 1_{0 \leq y_n \leq \theta}$$

$= 1$ if $\cancel{y_n \leq \theta}$ $y_i \leq \theta$ $\forall i$

otherwise $= 0$

~~hence max~~ $\cancel{y_n \leq \theta}$

$\frac{1}{\theta^n}$ is decreasing in θ hence

$\hat{\theta}$ = smallest value of θ s.t. $\theta \geq y_i$ y_i

$$\boxed{\therefore \hat{\theta} = \max(y_1, \dots, y_n)}$$

(5)

(a)

$$(1-aL)\,u_t = (1+bL)\,e_t$$

$\uparrow$ invertible?

• yes since root $1-aL=0$

$$L = \frac{1}{a} \qquad L > 0 \text{ if } |a|<1$$

$$u_t = \frac{1+bL}{1-aL}\,e_t$$

$$1+bL = (1-aL)(\delta_0 + \delta_1 L + \delta_2 L^2 + \delta_3 L^3 + \dots)$$

$$1 = \delta_0$$

$$bL = \delta_1 L - a\delta_0 L$$

$$\delta_1 = b + a$$

$$0L^2 = \delta_2 L^2 - a\delta_1 L^2 +$$

$$\delta_2 = a\delta_1 = a(a+b)$$

$$0L^3 = \delta_3 L^3 - a\delta_2 L^3$$

$$\delta_3 = a\delta_2 = a^2(a+b)$$

$$u_t = (1 + (a+b)L + a(a+b)L^2 + a^2(a+b)L^3 + \dots)\,e_t$$

$$\boxed{u_t = e_t + (a+b)\sum_{j=1}^{\infty} a^{j-1} e_{t-j}}$$

(b)

$$\mathbb{E}[u_t] = \mathbb{E}[e_t] + (a+b)\sum_{j=1}^{\infty} a^{j-1}\,\mathbb{E}[e_{t-j}] = 0$$

$$\cancel{\operatorname{var}[u_t] = \operatorname{var}\left(e_t + (a+b)\sum_{j=1}^{\infty} a^{j-1} e_{t-j}\right)}$$

$$\cancel{= \operatorname{var}(e_t) + (a+b)^2 \sum_{j=1}^{\infty} a^{(j-1)2} \operatorname{var}(e_{t-j})}$$

?

$$\mathrm{var}(u_t) = a^2\,\mathrm{var}(u_{t-1}) + \mathrm{var}(e_t) + b^2\,\mathrm{var}(e_{t-1}) + 2\mathrm{cov}(au_{t-1}, e_t) + 2\mathrm{cov}(au_{t-1}, be_{t-1}) + 2\mathrm{cov}(e_t, be_{t-1})$$

$$2\mathrm{cov}(au_{t-1}, e_t) = 2\mathrm{cov}(\cancel{u_{t-1}}\, a(au_{t-2} + e_{t-1} + be_{t-2}), e_t) = 0$$

$$2\mathrm{cov}(au_{t-1}, be_{t-1}) = 2\mathrm{cov}(a(au_{t-2} + e_{t-1} + be_{t-2}), be_{t-1}) = 2\mathrm{cov}(ae_{t-1}, be_{t-1}) = 2ab\,\mathrm{var}(e_{t-1})$$

$$2\mathrm{cov}(e_t, e_{t-1}) = 0$$

$$\mathrm{var}(u_t) = a^2\,\mathrm{var}(u_{t-1}) + \sigma_\varepsilon^2 + 2ab\,\sigma_\varepsilon^2 + b^2\sigma_\varepsilon^2$$

Stationary $\Rightarrow$ $\mathrm{var}(u_t) = \mathrm{var}(u_{t-1})$

$$\boxed{\mathrm{var}(u_t) = \frac{\sigma_\varepsilon^2(1 + 2ab + b^2)}{1 - a^2}}$$

$$\cancel{\mathrm{cov}(u_t, u_{t-h}) = \mathrm{cov}(au_{t-1} + e_t + be_{t-1}, \; au_{t-h-1} + e_{t-h} + be_{t-h-1})}$$

$$u_t \cdot u_{t-h} = au_{t-1}u_{t-h} + e_t u_{t-h} + be_{t-1}u_{t-h}$$

$$\mathbb{E}[u_t \cdot u_{t-h}] = a\,\mathbb{E}[u_{t-1}u_{t-h}] + \mathbb{E}[e_t u_{t-h}] + b\,\mathbb{E}[e_{t-1}u_{t-h}]$$

$$\gamma_u(h) = a\,\gamma_u(h-1) + \mathbb{E}[e_t u_{t-h}] + b\,\mathbb{E}[e_{t-1}u_{t-h}]$$

$h=1$: $\gamma_u(1) = a\,\gamma_u(0) + \mathbb{E}[e_t u_{t-1}] + b\,\mathbb{E}[e_{t-1}u_{t-1}]$

$$\mathbb{E}\Big[\underbrace{e_t e_{t-1}}_{=0} + (a+b)\underbrace{\sum_{j=1}^{\infty} a^{j-1} e_{t-1-j}\, e_t}_{=0}\Big] \qquad \mathbb{E}\Big[\underbrace{e_{t-1}e_{t-1}}_{=\sigma_\varepsilon^2} + (a+b)\underbrace{\sum_{j=1}^{\infty} a^{j-1} u_{t-1-j}\, e_{t-1}}_{=0}\Big]$$

$$\boxed{\gamma_u(1) = a\,\gamma_u(0) + b\,\sigma_\varepsilon^2}$$

$h \geq 2$: $\boxed{\gamma_u(h) = a\,\gamma_u(h-1)}$

$h=1$: $\quad cov(u_t, u_{t-h}) = \dfrac{a\,\sigma_\varepsilon^2\,(1+2ab+b^2)}{1-a^2} + b\sigma_\varepsilon^2$

$$= \frac{\sigma_\varepsilon^2\left[(a+2a^2b+ab^2)+b(1-a^2)\right]}{1-a^2}$$

$$= \frac{\sigma_\varepsilon^2\left[a+\cancel{2}a^2b+ab^2+b-\cancel{a^2b}\right]}{1-a^2}$$

$$= \frac{\sigma_\varepsilon^2\left[a+a^2b+ab^2+b\right]}{1-a^2}$$

$h \geq 2$: $\quad cov(u_t, u_{t-h}) = a \cdot cov(u_t, u_{t-h-1})$

(c)

$$y_t = \mu + e_t + (a+b)\sum_{j=1}^{\infty} a^{j-1} e_{t-j}$$

$$\sum_{j=0}^{\infty}|\psi_j| = 1 + |(a+b)| + |a(a+b)| + |a^2(a+b)| + \cdots$$

$$= 1 + |a+b| + |a^2+ab| + |a^3+a^2b| + \cdots$$

$$\sum_{j=0}^{\infty}|(a+b)a^j| = \left|\frac{a+b}{1-a}\right| + 1 < \infty$$

$\therefore$ holds.
(for $|a|<1$ and $|b|<1$)

(d)

$$\bar{y}_T = \frac{1}{T}\sum_{t=1}^{T} y_t \qquad y_t = \mu + e_t + (a+b)\sum_{j=1}^{\infty} a^{j-1} e_{t-j}$$

$$\sqrt{T}\,(\bar{y}_T - \mu) \xrightarrow{d} N\left(0, \sum_{h=-\infty}^{\infty} \gamma_y(h)\right)$$

$$\cancel{\gamma_y(h) = a\,\gamma_y(h-1)}$$

$$h=2 \ : \ \cancel{\gamma_y(2) = a\,\sigma_\varepsilon^2 \frac{(a + a^2 b + ab^2 + b)}{(1-a^2)}}$$

$$\cancel{\sum_{h=-\infty}^{\infty} |\gamma_j| = \sigma_\varepsilon^2 \psi^2(1)}$$

$$\cancel{= \sigma_\varepsilon^2 \left(1 + (a+b) + a(a+b) + a^2(a+b) + \cdots\right)}$$

$$y_t = \mu + \frac{1+bL}{1-aL} e_t$$

$$\psi(L) = \frac{1+bL}{1-aL} \qquad \psi^2(1) = \left(\frac{1+b}{1-a}\right)^2$$

$\therefore$

$$\boxed{\sqrt{T}\,(\bar{y}_T - \mu) \xrightarrow{d} N\left(0, \sigma_\varepsilon^2 \left(\frac{1+b}{1-a}\right)^2\right)}$$

(6)

(a)

$$y_t = \lambda + y_{t-1} + v_t$$

$$y_{t-1} = \lambda + y_{t-2} + v_{t-1}$$

$$y_t = 2\lambda + y_{t-2} + v_t + v_{t-1}$$

$$y_t = h \cdot \lambda + y_{t-h} + \sum_{j=0}^{h-1} v_{t-j}$$

$h = t$

$$y_t = t\lambda + y_0 + \sum_{j=0}^{t-1} v_{t-j} = \underline{t \cdot \lambda + \sum_{j=1}^{t} v_j} \qquad (y_0 = 0)$$

$$\mathbb{E}[y_t] = t \cdot \lambda + \sum_{j=1}^{t} \mathbb{E}[v_j] = \underline{\lambda \cdot t}$$

$$\mathrm{Var}(y_t) = \mathrm{var}\left(\sum_{j=1}^{t} v_j\right) = \sum_{j=1}^{t} \mathrm{var}(v_j) = \underline{t \cdot \sigma_v^2}$$

$$(\mathrm{cov}(v_i, v_j) = 0 \quad \forall\ i \neq j)$$

$$\mathrm{cov}(y_s ; y_t) = \mathrm{cov}\left(\cancel{t \cdot \lambda} + \sum_{j=1}^{t} v_j\ ,\ \cancel{s \cdot \lambda} + \sum_{i=1}^{s} v_i\right)$$

$$= \mathrm{cov}\left(\sum_{j=1}^{t} v_j\ ,\ \sum_{i=1}^{s} v_i\right) = \sum_{j=1}^{t} \sum_{i=1}^{s} \mathrm{cov}(v_j, v_i)$$

$$\mathrm{cov}(v_j, v_i) = 0 \quad \forall\ i \neq j \qquad =$$

$$= \min\{t, s\}\ \sigma_v^2 = s\sigma_v^2$$

(b)

(c) ?

(d)

(7)

(a)

$$\mathbb{E}[y_t] = \delta t^3 + \mathbb{E}[u_t] = \delta t^3$$

$$\text{var}(y_t) = \mathbb{E}\left[(y_t - \mathbb{E}[y_t])^2\right] = \mathbb{E}[u_t^2] = \sigma_u^2$$

$$\text{cov}(y_t, y_s) = \mathbb{E}\left[(y_t - \mathbb{E}[y_t])(y_s - \mathbb{E}[y_s])\right] = \mathbb{E}[u_t u_s]$$

$$= \min\{s, t\}\, \sigma_u^2$$

$$= s\, \sigma_u^2$$

(b) OLS

$$\text{argmin} \sum_{t=1}^{T} (y_t - \delta t^3)^2$$

foc:

$$0 = \cancel{-2} \sum_{t=1}^{T} t^3 (y_t - \hat{\delta} t^3)$$

$$0 = \sum_{t=1}^{T} t^3 y_t - \hat{\delta} \sum_{t=1}^{T} t^6$$

$$\hat{\delta} = \frac{\sum_{t=1}^{T} t^3 y_t}{\sum_{t=1}^{T} t^6}$$

$$\hat{\delta} = \frac{\sum_{t=1}^{T} t^3 (\delta t^3 + u_t)}{\sum_{t=1}^{T} t^6} = \delta + \frac{\sum_{t=1}^{T} t^3 \cdot u_t}{\sum_{t=1}^{T} t^6}$$

$$\hat{\delta} = \delta + \frac{\sum_{t=1}^{T} t^3 u_t}{\sum_{t=1}^{T} t^6} \qquad \mathbb{E}[\hat{\delta}] = \delta + \frac{\sum_{t=1}^{T} t^3 \overbrace{\mathbb{E}[u_t]}^{=0}}{\sum_{t=1}^{T} t^6} = \delta$$

$$\hat{\delta} - \delta = \frac{\sum_{t=1}^{T} t^3 u_t}{\sum_{t=1}^{T} t^6}$$

$$\text{var}(\hat{\delta}) = \text{var}\left[\frac{\sum_{t=1}^{T} t^3 u_t}{\sum_{t=1}^{T} t^6}\right] = \frac{1}{\left[\sum_{t=1}^{T} t^6\right]^2} \sum_{t=1}^{T} t^{3\cdot 2} \underbrace{\text{var}(u_t)}_{=\sigma_u^2}$$

$$= \frac{\sigma_u^2}{\sum_{t=1}^{T} t^6}$$

$$\sum_{t=1}^{T} t^6 = \frac{1}{42} T(T+1)(2T+1)(3T^4 + 6T^3 - 3T + 1)$$

$$\text{var}(\hat{\delta}) = \frac{\sigma_u^2}{\frac{1}{42} T(T+1)(2T+1)(3T^4 + 6T^3 - 3T + 1)} \longrightarrow 0 \text{ as } T \text{ gets large}$$

(tends to 0 fast)

OR:

$$\text{var}(\hat{\delta}) = \frac{\frac{1}{T^7}\sigma_u^2}{\frac{1}{T^7}\sum_{t=1}^{T} t^6} = \frac{\frac{1}{T^7}\sigma_u^2}{\frac{1}{7}}$$

$$\text{var}(\hat{\delta}) = 7\frac{\sigma_u^2}{T^7} \rightarrow 0 \text{ very quickly!}$$

∴ since mean square convergence implies convergence in probability.

$$\hat{\delta} - \delta \xrightarrow{p} 0 \qquad \hat{\delta} \xrightarrow{p} \delta \quad \therefore \text{ consistent.}$$

(c)

$$\hat{\delta} = \delta + \frac{\sum_{t=1}^{T} t^3 u_t}{\sum_{t=1}^{T} t^6} \qquad \hat{\delta} - \delta = \frac{\sum_{t=1}^{T} t^3 u_t}{\sum_{t=1}^{T} t^6}$$

recall :

$$\frac{1}{T^{v+1}} \sum_{t=1}^{T} t^v \longrightarrow \int_0^1 r^v dr = \frac{1}{v+1}$$

$$\sqrt{T}\, T^3 (\hat{\delta} - \delta) = \frac{\sqrt{T} \frac{1}{T^4} \sum_{t=1}^{T} t^3 u_t}{\frac{1}{T^7} \sum_{t=1}^{T} t^6} \qquad \frac{1}{T^7} \sum_{t=1}^{T} t^6 \longrightarrow \frac{1}{7}$$

$$\sqrt{T} \frac{1}{T^4} \sum_{t=1}^{T} t^3 u_t = \sqrt{T} \sum_{t=1}^{T} \left(\frac{t^3}{T^4}\right) u_t$$

mds = m_t.

$$S_T^2 = \sum_{t=1}^{T} \mathbb{E}\left[\left(\frac{t^3}{T^4}\right)^2 u_t^2\right] = \sum_{t=1}^{T} t^6/T^8\, \mathbb{E}[u_t^2] = \sigma_u^2 \sum_{t=1}^{T} t^6/T^8$$

Conditions:

?

(i) $$\sum_{t=1}^{T} m_t^2/S_T^2 = \sum_{t=1}^{T} \frac{t^6/T^8\, u_t^2}{\sigma_u^2 \sum_{t=1}^{T} t^6/T^8} = \frac{\sum_{t=1}^{T} t^6/T^8\, u_t^2}{\sigma_u^2 \sum_{t=1}^{T} t^6/T^8}$$

$$\frac{1}{T}\frac{1}{T^7} \sum_{t=1}^{T} t^6 \longrightarrow \frac{1}{T}\frac{1}{7} \qquad \frac{\cancel{\frac{1}{T}} \sum_{t=1}^{T} u_t^2}{\cancel{\frac{1}{T}}\, \sigma_u^2} \xrightarrow{p} 1$$

(ii') $$\sum_{t=1}^{T} \mathbb{E}\left| m_t / s_T \right|^{2+\delta} = \sum_{t=1}^{T} \mathbb{E}\left[\frac{\frac{t^3}{T^4} u_t}{\sigma_u^2 \sum_{t=1}^{T} \frac{t^6}{T^8}} \right]$$

$$= \sum_{t=1}^{T} \frac{\left(\frac{t^3}{T^4}\right)^4 \mathbb{E} u_t^4}{s_T^4} = \frac{\mu_4}{s_T^4} \sum_{t=1}^{T} \left(\frac{t^3}{T^4}\right)^4$$

$$= \frac{\mu_4}{\underbrace{s_T^4 / T}_{\to \infty}} \frac{1}{T} \underbrace{\sum_{t=1}^{T} \left(\frac{t^3}{T^4}\right)^4}_{\to \text{constant}} \quad \therefore \xrightarrow{p} 0$$

(c) $$\hat{\delta} - \delta = \frac{\sum_{t=1}^{T} t^3 u_t.}{\sum_{t=1}^{T} t^6}$$

recall $$\frac{1}{T^{v+1}} \sum_{t=1}^{T} t^v \longrightarrow \int_0^1 r^v dr = \frac{1}{v+1}$$

$$T^7 (\hat{\delta} - \delta) = \frac{\sum_{t=1}^{T} t^3 u_t}{T^{-7} \sum_{t=1}^{T} t^6} \qquad T^{-7} \sum_{t=1}^{T} t^6 \longrightarrow \int_0^1 r^6 dr = \frac{1}{7}$$

$$\sum_{t=1}^{T} t^3 u_t.$$

8

(a)

Stable if roots of $1 - 0.7L + 0.1L^2$ are outside the unit circle.

roots : $L = 5$ and $L = 2$ ∴ outside ∴ Stable.

(b)

$$5L = (1 - 0.7L + 0.1L^2)(\delta_0 + \delta_1 L + \delta_2 L^2 + \delta_3 L^3 + \cdots)$$

$$0 = \delta_0$$

$$5L = \delta_1 L - \delta_0 0.7L$$

$$\delta_1 = 5$$

$$0L^2 = \delta_2 L^2 - 0.7\delta_1 L^2 + \delta_0 0.1L^2$$

$$\delta_2 = 0.7 \cdot 5 = 3.5$$

$$0L^3 = \delta_3 L^3 - 0.7\delta_2 L^3 + 0.1\delta_1 L^3$$

$$\delta_3 = 0.7 \cdot 3.5 \mp 0.1 \times 5$$

$$\delta_3 = 1.95$$

2nd lag multiplier = $\boxed{\delta_2 = 3.5}$

total multiplier = $\dfrac{B_r(1)}{C_p(1)} = \dfrac{5}{1-0.7+0.1}$ $\boxed{= 12.5}$

~~Mean lag~~ $= \dfrac{\beta}{\beta} - \dfrac{1-\gamma_1 - 2\cdot\gamma_2}{1-\gamma_1-\gamma_2} = \dfrac{5}{5} - \dfrac{1-0.7+\cancel{2\cdot 0.1}}{1-0.7+0.1} = \cancel{\boxed{-\frac{1}{4}}}\ \boxed{\frac{11}{6}}$

~~Median lag~~ =

$$\text{Mean lag} = \frac{\beta}{\beta} - \frac{-\gamma_1 - 2\gamma_2}{1-\gamma_1-\gamma_2} = \frac{5}{5} + \frac{\gamma_1 + 2\gamma_2}{1-\gamma_1-\gamma_2}$$

$$= 1 + \frac{0.7 - 2\cdot 0.1}{1 - 0.7 + 0.1}$$

$$\boxed{= 2.25}$$

$$\text{Median lag} = \min_q \sum_{j=0}^{q} \delta_j \geq 0.5 \times 2.25 = \frac{9}{8} = 1.125$$

$$\text{median lag} = 1$$

(c)

$$y_t - \gamma_1 y_{t-1} - \gamma_2 y_{t-2} = \beta x_t + u_t.$$

$$y_t - y_{t-1} = (\gamma_1 - 1) y_{t-1} + \gamma_2 y_{t-2} + \beta x_t + u_t.$$

$$\Delta y_t = (\gamma_1 - 1) y_{t-1} - \gamma_2 (y_{t-1} - y_{t-2}) + \gamma_2 y_{t-1} + \beta (x_t - x_{t-1}) + \beta x_{t-1} + u_t.$$

$$\Delta y_t = (\gamma_2 + \gamma_1 - 1) y_{t-1} + \beta x_{t-1} - \gamma_2 \Delta y_{t-1} + \beta \Delta x_t + u_t.$$

$$\Delta y_t = (\gamma_2 + \gamma_1 - 1)\left[y_{t-1} - \frac{-\beta}{(\gamma_2 + \gamma_1 - 1)} x_{t-1} \right] - \gamma_2 \Delta y_{t-1} + \beta \Delta x_t + u_t$$

LR eq.: $\mathbb{E} y_{t-1} = \dfrac{\beta}{1-\gamma_1-\gamma_2} \mathbb{E}[x_{t-1}]$

speed of adjustment $= (\gamma_2 + \gamma_1 - 1)$

$$\mathbb{E}[y_t] = \frac{\beta}{1-\gamma_1-\gamma_2} \mathbb{E}[x_t]$$

Econometrics 2019

(1)

(a) $y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + u_i$

$$Y = X\beta + U$$

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \quad X = \begin{pmatrix} x_{11} & x_{21} & x_{31} & x_{41} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1n} & x_{2n} & x_{3n} & x_{4n} \end{pmatrix} \quad \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_4 \end{pmatrix} \quad U = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} (Y - X\beta)'(Y - X\beta)$$

$$= \underset{\beta}{\operatorname{argmin}} (Y' - \beta'X')(Y - X\beta)$$

$$= \operatorname{argmin} (Y'Y - Y'X\beta - \beta'X'Y + \beta'X'X\beta)$$

$$= \operatorname{argmin} (Y'Y - 2Y'X\beta + \beta'X'X\beta)$$

foc: $$\frac{\partial}{\partial \beta} = (-2Y'X)' + (X'X + (X'X)')\hat{\beta} = 0$$

$$\left(\frac{\partial (Ax)}{\partial x} = A' \ , \ \frac{\partial (x'Ax)}{\partial x} = (A + A')x \right)$$

$$-\cancel{2}X'Y + \cancel{2}X'X\hat{\beta} = 0$$

$$\hat{\beta} = (X'X)^{-1}X'Y$$

$$\hat{\beta} = (X'X)^{-1}X'(X\beta + U)$$

$$\boxed{\hat{\beta} = \beta + (X'X)^{-1}X'U}$$

$$\boxed{\begin{pmatrix}\hat\beta_1\\ \hat\beta_2\\ \hat\beta_3\\ \hat\beta_4\end{pmatrix} = \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\end{pmatrix} + \begin{bmatrix}\sum_{i=1}^n x_{1i}^2 & \cdots & \sum_{i=1}^n x_{1i}x_{4i}\\ \vdots & \ddots & \vdots\\ \sum_{i=1}^n x_{4i}x_{1i} & \cdots & \sum_{i=1}^n x_{4i}^2\end{bmatrix}^{-1} \begin{bmatrix}\sum_{i=1}^n x_{1i}\cdot u_i\\ \vdots\\ \sum_{i=1}^n x_{4i}u_i\end{bmatrix}}$$

$$X'X = \underbrace{\underset{\substack{\cancel{n\times 4}\\ 4\times n}}{\begin{pmatrix}x_{11} & \cdots & x_{1n}\\ x_{21} & \cdots & x_{2n}\\ x_{31} & \cdots & x_{3n}\\ x_{41} & \cdots & x_{4n}\end{pmatrix}} \underset{\substack{\cancel{4\times n}\\ n\times 4}}{\begin{pmatrix}x_{11} & x_{21} & x_{31} & x_{41}\\ \vdots & \vdots & \vdots & \vdots\\ x_{1n} & x_{2n} & x_{3n} & x_{4n}\end{pmatrix}}}_{4\times 4} = \begin{bmatrix}\sum_{i=1}^n x_{1i}^2 & \sum_{i=1}^n x_{1i}x_{2i} & \sum_{i=1}^n x_{1i}x_{3i} & \sum_{i=1}^n x_{1i}x_{4i}\\ \sum_{i=1}^n x_{2i}x_{1i} & \sum_{i=1}^n x_{2i}x_{2i} & \sum_{i=1}^n x_{2i}x_{3i} & \sum_{i=1}^n x_{2i}x_{4i}\\ \sum_{i=1}^n x_{3i}x_{1i} & \sum_{i=1}^n x_{3i}x_{2i} & \sum_{i=1}^n x_{3i}^2 & \sum_{i=1}^n x_{3i}x_{4i}\\ \sum_{i=1}^n x_{4i}x_{1i} & \sum_{i=1}^n x_{4i}x_{2i} & \sum_{i=1}^n x_{4i}x_{3i} & \sum_{i=1}^n x_{4i}^2\end{bmatrix}$$

$$X'u = \underbrace{\underset{4\times n}{\begin{pmatrix}x_{11} & \cdots & x_{1n}\\ \vdots & \ddots & \vdots\\ x_{41} & \cdots & x_{4n}\end{pmatrix}} \underset{n\times 1}{\begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}}}_{4\times 1} = \begin{bmatrix}\sum_{i=1}^n x_{1i}\cdot u_i\\ \vdots\\ \sum_{i=1}^n x_{4i}u_i\end{bmatrix}$$

(b) Consistency:

$$\hat\beta = \beta + \left[\frac{X'X}{n}\right]^{-1}\frac{X'u}{n}$$

$$\frac{X'u}{n} = \begin{bmatrix}\frac{1}{n}\sum_{i=1}^n x_{1i}u_i\\ \vdots\\ \frac{1}{n}\sum_{i=1}^n x_{4i}u_i\end{bmatrix} = n^{-1}\sum_{i=1}^n X_i u_i \qquad \text{where } X_i = \begin{pmatrix}x_{1i}\\ \vdots\\ x_{4i}\end{pmatrix}$$

$$\mathbb{E}[X_i u_i] = \cancel{\mathbb{E}[\mathbb{E}[X_i u_i]X_i} \; \mathbb{E}[X_i]\,\mathbb{E}[u_i] = 0$$

(independence

$$\mathrm{var}(X_i u_i) = \mathbb{E}[X_i^2 u_i^2] - \mathbb{E}[X_i]\underbrace{\mathbb{E}[u_i]}_{=0}$$

$$\mathbb{E}[X_i^2] = \mathrm{var}(X_i) - \mathbb{E}[X_i] \longrightarrow \quad = \underbrace{\mathbb{E}[X_i^2]\,\mathbb{E}[u_i^2]}_{\text{(independence)}} = \cancel{8\text{-}(1-8)}\underset{(9)}{}\cdot(0) = 0$$

$$= 1 + 8$$

$$= -\cancel{17}.9$$

hence

$$\frac{X'u}{n} \xrightarrow{p} 0$$

$$\frac{X'X}{n} = \cancel{\sum_{i=1}^{4} X_i' X_i \quad X_i =}$$

$$X_i = \begin{pmatrix} x_{1i} \\ \vdots \\ x_{4i} \end{pmatrix}$$

$$\left(\frac{X'X}{n}\right)^{-1} \xrightarrow{p} \left(\mathbb{E}[X_i X_i']\right)^{-1}$$

$$\mathbb{E}\left[\underbrace{\begin{pmatrix} x_{1i} \\ \vdots \\ x_{4i} \end{pmatrix}}_{4\times 1} \underbrace{(x_{1i} \cdots x_{4i})}_{1\times 4}\right] = \mathbb{E}\left[\begin{pmatrix} x_{1i}^2 & x_{1i}x_{2i} & x_{1i}x_{3i} & x_{1i}x_{4i} \\ \vdots & & & \vdots \\ x_{4i}x_{1i} & \cdots & \cdots & x_{4i}^2 \end{pmatrix}\right]$$

$$= \begin{matrix} \mathbb{E}[x_{1i}^2] & \mathbb{E}[x_{1i}]\mathbb{E}[x_{2i}] & \cdots & \mathbb{E}[x_{1i}]\mathbb{E}[x_{4i}] \\ \mathbb{E}[x_{2i}]\mathbb{E}[x_{3i}] & \ddots & & \vdots \\ \vdots & & \cdots & \\ \mathbb{E}[x_{4i}]\mathbb{E}[x_{1i}] & \cdots & & \mathbb{E}[x_{4i}^2] \end{matrix}$$

$$= \begin{bmatrix} 9 & 16 & 16 & 16 \\ 16 & 9 & 16 & 16 \\ 16 & 16 & 9 & 16 \\ 16 & 16 & 16 & 9 \end{bmatrix}^{-1} = \cancel{\text{definit}} \text{ invertible}$$

(c)

Power of test. :

let $\beta =$ probability of not rejecting null when alternative is true

(Type II error)

$$\boxed{\text{Power} = 1-\beta}$$

(d)

F-test. (using iid normal errors)

$$F = \frac{(R\hat{\beta}-q)' \left\{ \hat{\sigma}_u^2 R (X'X)^{-1} R' \right\}^{-1} (R\hat{\beta}-q)}{J} \xrightarrow{D} F_{J,\, n-K}$$

$$J = 2 \qquad \hat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_4 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix} \qquad q = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$F = \frac{\left[\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_4 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} \right] \left\{ (1) \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} \sum_{i=1}^{n} x_{1i}^2 & \cdots & \sum_{i=1}^{n} x_{1i}x_{4i} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} x_{4i}x_{1i} & \cdots & \sum_{i=1}^{n} x_{4i}^2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 3 & 0 \\ 0 & 1 \\ 0 & -2 \end{pmatrix} \right\}^{-1} \left[\cdots \right]}{2}$$

$$\boxed{F_{2,\, n-4}}$$

reject if $F > CV_{\alpha}$.

(e)

$$R^2 = \frac{ESS}{TSS} = \frac{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}$$

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i \qquad \bar{y} = \beta_0 + \beta_1 \bar{x} + \bar{u}$$

$$\hat{y}_i - \bar{y} = (\hat{\beta}_0 - \beta_0) + \hat{\beta}_1 x_i - \beta_1 \bar{x} - \bar{u}$$

$$\hat{\beta}_0 = \beta_0 + \bar{u} - (\hat{\beta}_1 - \beta_1)\bar{x}$$

$$\hat{\beta}_0 - \beta_0 = \bar{u} - (\hat{\beta}_1 - \beta_1)\bar{x}$$

$$\hat{y}_i - \bar{y} = \cancel{\bar{u}} - \hat{\beta}_1 \bar{x} + \cancel{\beta_1 \bar{x}} + \hat{\beta}_1 x_i - \cancel{\beta_1 \bar{x}} - \cancel{\bar{u}}$$

$$= \hat{\beta}_1 (x_i - \bar{x})$$

$$R^2 = \frac{\hat{\beta}_1^2 \sum_{i=1}^{n}(x_i - \bar{x})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2} \qquad \hat{\beta}_1 = \frac{\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$R^2 = \frac{\dfrac{\left[\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x})\right]^2}{\left[\sum_{i=1}^{n}(x_i - \bar{x})^2\right]^{\cancel{2}}} \cancel{\sum_{i=1}^{n}(x_i - \bar{x})^2}}{\sum_{i=1}^{n}(y_i - \bar{y})^2}$$

$$R^2 = \frac{\left[\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x})\right]^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2} = \underline{\underline{\frac{\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}}}$$

(f)

① linear regression is best linear approximation of conditional expectation.

② If conditional expectation is linear then lin. regression ~~is best approximation~~ coincides with it ~~of it of any function.~~

① regression solves : $\min_{b_0, b_1} \mathbb{E}\left[\{Y - (b_0 + b_1 X)\}^2\right]$

② CEF solves : $\min_{m} \mathbb{E}\left[\{Y - m(X)\}\right]$

and when CEF is linear CEF & linear regression coincide.

when CEF $\neq$ linear then regression is best linear approximation of it.

(g)

$$f(x,y) = (x+y)\, I_{[0,1]}(x)\, I_{[0,1]}(y)$$

$$\boxed{f(y|x) = \frac{f(x,y)}{f(x)}}$$

$$f(x) = \int_0^1 f(x,y)\,dy = \int_0^1 (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_0^1$$

$$\left(x + \frac{1}{2}\right) - \left(0x + \frac{0^2}{2}\right)$$

$$f(x) = x + \tfrac{1}{2}$$

$$f(y|x) = \frac{x+y}{x+\frac{1}{2}}\, I_{[0,1]}(x)\, I_{[0,1]}(y)$$

$$\int_0^1 y \frac{f(x,y)}{f(x)}\,dy = \int_0^1 y\, f(y|x)\,dy$$ to give $\mathbb{E}[Y|X=x]$

$$= \int_0^1 y\, \frac{x+y}{x+\frac{1}{2}}\,dy$$

$$= \int_0^1 \frac{xy}{x+\frac{1}{2}} + \frac{y^2}{x+\frac{1}{2}}\,dy$$

$$= \left[\frac{xy^2}{2(x+\frac{1}{2})} + \frac{y^3}{3(x+\frac{1}{2})}\right]_0^1 = \underline{\underline{\frac{x}{2(x+\frac{1}{2})} + \frac{1}{3(x+\frac{1}{2})}}}$$

$$= \underline{\underline{\frac{3x+2}{6x+3}}}$$

[Note: $\displaystyle\int_{-\infty}^{\infty} (x+y)\, I_{[0,1]}(x) I_{[0,1]}(y)\,dy = \int_0^1 (x+y)\,dy$]

(2)

(a)

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \qquad \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \qquad X = \begin{pmatrix} 1 & x_{11} \\ 1 & x_{12} \\ \vdots & \vdots \\ 1 & x_{1n} \end{pmatrix} \qquad U = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

$$\hat{\beta} = \beta + (X'X)^{-1} X' U$$

$$\begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} + \left[\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_{11} & x_{12} & \cdots & x_{1n} \end{pmatrix} \begin{pmatrix} 1 & x_{11} \\ \vdots & \vdots \\ 1 & x_{1n} \end{pmatrix} \right]^{-1} \begin{pmatrix} 1 & \cdots & 1 \\ x_{11} & \cdots & x_{1n} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

$$\cancel{\hat{\beta} - \beta = \left(\frac{X'X}{n}\right)^{-1} \frac{X'U}{n}}$$

$$\cancel{\left(\frac{X'X}{n}\right)^{-1}}$$

$$\mathbb{E}[\hat{\beta}] = \beta + \mathbb{E}\left[(X'X)^{-1} X' U\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[(X'X)^{-1} X' U \mid X\right]\right]$$

$$= \mathbb{E}\left[(X'X)^{-1} X' \, \mathbb{E}[U \mid X]\right]$$

$$\mathbb{E}[U \mid X] = \begin{pmatrix} \gamma + \delta x_{11} \\ \vdots \\ \gamma + \delta x_{1n} \end{pmatrix} \neq 0$$

hence

$$\mathbb{E}\left[(X'X)^{-1} X' \, \mathbb{E}[U \mid X]\right] \neq 0$$

$$X' \mathbb{E}[u|X] = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_{11} & x_{12} & \cdots & x_{1n} \end{pmatrix} \begin{pmatrix} \gamma + \delta x_{11} \\ \vdots \\ \gamma + \delta x_{1n} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{n} (\gamma + \delta x_{1i}) \\ \sum_{i=1}^{n} (\gamma x_{1i} + \delta x_{1i}^2) \end{pmatrix}$$

$\gamma \neq 0$, $\delta \neq 0$
hence this matrix is not zero.

(b)

$Z_i = (1_{4} \;\; z_{1i})$ $\quad \mathbb{E}[u_i \mid z_i] = 0$

(i) other assumptions?

Relevance: instrument is correlated with endogeneous independent variable.

$$\text{Cov}(z_i, x_i) \neq 0$$

this implies $Z'X$ is invertible.

(ii)

$$\frac{X'Z}{n} = \underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_{11} & x_{12} & \cdots & x_{1n} \end{pmatrix}}_{2 \times n} \underbrace{\begin{pmatrix} 1 & z_{11} \\ \vdots & \vdots \\ 1 & z_{1n} \end{pmatrix}}_{n \times 2} = \begin{pmatrix} \sum_{i=1}^{n} 1 = n & \sum_{i=1}^{n} z_{1i} \\ \sum_{i=1}^{n} x_{1i} & \sum_{i=1}^{n} z_{1i} x_{1i} \end{pmatrix}$$

2×2

$$\frac{X'Z}{n} = \begin{bmatrix} 1 & \frac{1}{n} \sum_{i=1}^{n} z_{1i} \\ \frac{1}{n} \sum_{i=1}^{n} x_{1i} & \frac{1}{n} \sum_{i=1}^{n} z_{1i} x_{1i} \end{bmatrix}$$

- hence 2×2 matrix.
- rank 2
  since z_{1i}, x_{1i} not independent?
  (rank L where L = no. of instruments = 2 here).

(iii)

$$\mathbb{E}[u_i | z_i] = 0$$

$$\mathbb{E}[z_i u_i] = \mathbb{E}\left[\mathbb{E}[z_i u_i | z_i]\right] = \mathbb{E}\left[z_i \mathbb{E}[u_i | z_i]\right] = 0$$

$$\mathbb{E}[z_i u_i] = 0$$

$$Z'y = Z'(X\beta + u)$$

$$= Z'X\beta + Z'u$$

$$Z'u = \begin{pmatrix} 1 & \cdots & 1 \\ z_{11} & \cdots & z_{1n} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{n} u_i \\ \sum_{i=1}^{n} u_i z_{1i} \end{pmatrix}$$

$$\frac{Z'y}{n} = \frac{Z'X}{n}\beta + \frac{Z'u}{n}$$

$$\frac{Z'u}{n} = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} u_i \\ \frac{1}{n}\sum_{i=1}^{n} u_i z_{1i} \end{pmatrix} \xrightarrow{p} \begin{pmatrix} \mathbb{E}[u_i] \\ \mathbb{E}[u_i z_{1i}] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by iid LLN.

$$\frac{Z'y}{n} = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} y_i \\ \frac{1}{n}\sum_{i=1}^{n} y_i z_{1i} \end{pmatrix}$$

$$\frac{Z'X}{n} = \begin{bmatrix} 1 & \frac{1}{n}\sum_{i=1}^{n} x_{1i} \\ \frac{1}{n}\sum_{i=1}^{n} z_{1i} & \frac{1}{n}\sum_{i=1}^{n} z_{1i} x_{1i} \end{bmatrix}$$

$\frac{Z'X}{n}$ is 2×2 rank 2 (square + full rank) hence invertible.

$$\beta = \left(\frac{Z'X}{n}\right)^{-1} \frac{Z'Y}{n}$$

$$\hat{\beta}_{IV} = \left(\frac{Z'X}{n}\right)^{-1} \frac{Z'Y}{n}$$

(iv)

As before but use instrument:

$$Z_i = (1 \quad z_{1i} \quad x_{2i})$$

$$Z = \begin{pmatrix} 1 & z_{11} & x_{21} \\ 1 & z_{12} & x_{22} \\ \vdots & \vdots & \vdots \\ 1 & z_{1n} & x_{2n} \end{pmatrix}$$

and regress as before with.

$$\hat{\beta}_{IV} = \left(\frac{Z'X}{n}\right)^{-1} \frac{Z'Y}{n}$$

(4)

(c)

(i) $Z_i = (1 \quad z_{1i} \quad z_{2i})$

$$Z = \begin{pmatrix} 1 & z_{11} & z_{21} \\ \vdots & \vdots & \vdots \\ 1 & z_{1n} & z_{2n} \end{pmatrix} \quad n \times 3$$

new $Z'X$ is a 3×2 matrix with rank 2

$\Rightarrow$ not invertible, hence ILS fails.

- 2SLS:

① $X = Z\alpha + V$

$\hat{X} = Z\hat{\alpha}$

② $Y = \hat{X}\beta + u$

$$\hat{\alpha} = (Z'Z)^{-1} Z'X$$

$$\hat{X} = Z(Z'Z)^{-1} Z'X$$

$$\hat{\beta}_{2SLS} = (\hat{X}'\hat{X})^{-1} (\hat{X}'Y)$$

$$= \{X'Z\cancel{(Z'Z)^{-1}Z'Z}(Z'Z)^{-1}Z'X\}^{-1} \cancel{Z(Z'Z)^{-1}Z'XY} \; (X'Z(Z'Z)^{-1}Z'Y)$$

$$\boxed{\hat{\beta}_{2SLS} = \{X'Z(Z'Z)^{-1}Z'X\}^{-1} X'Z(Z'Z)^{-1}Z'Y}$$

(ii)

$$\min_b \; (y - Xb)' Z \Omega Z' (y - Xb)$$

$$\min_b \; (y' - b'X') Z \Omega Z' (y - Xb)$$

$$\min_b \; (y'Z\Omega Z'y + b'X'Z\Omega Z'Xb - y'Z\Omega Z'Xb - b'X'Z\Omega Z'y)$$

transposes of eachother

$$\min_b \; (y'Z\Omega Z'y - 2\, y'Z\Omega Z'Xb + b'X'Z\Omega Z'Xb)$$

foc:

$$-2\,(y'Z\Omega Z'X)' + (X'Z\Omega Z'X + (X'Z\Omega Z'X)')\,\hat{b} = 0$$

$$-\cancel{2}\,(X'Z\Omega Z'y) + \cancel{2}\,(X'Z\Omega Z'X)\,\hat{b} = 0$$

$$\cancel{\hat{b} = (X'Z\Omega Z'y)\,(}$$

$$\boxed{\hat{b} = \{X'Z\Omega Z'X\}^{-1} (X'Z\Omega Z'y)}$$

(assumed $\Omega = \Omega'$ square symmetrical)

(iii) $\Omega = (Z'Z)^{-1}$

$$\hat{b} = (X'Z(Z'Z)^{-1}Z'X)^{-1}(X'Z(Z'Z)^{-1}Z'Y)$$

$= 2SLS$ estimator.

3.

(a)

$$y_i^* = x_i'\beta + u_i$$

$$P(y_i = 1) = P(y_i^* > 0)$$

$$= P(x_i'\beta + u_i > 0)$$

$$= P(u_i > -x_i'\beta)$$

$$u_i \sim N(0, \sigma^2)$$

$$Z = \frac{u_i - 0}{\sigma}$$

$$= P\left(\frac{u_i - 0}{\sigma} > \frac{-x_i'\beta - 0}{\sigma}\right)$$

$$\Phi(x) = P(X \leq x)$$

$$= 1 - P\left(Z \leq \frac{-x_i'\beta}{\sigma}\right)$$

$$= 1 - P\left(Z \geq \frac{x_i'\beta}{\sigma}\right)$$

$$= P\left(Z \leq \frac{x_i'\beta}{\sigma}\right) = \boxed{\Phi\left(\frac{x_i'\beta}{\sigma}\right)}$$

$$P(y_i = 0) = 1 - P(y_i = 1)$$

$$\boxed{P(y_i = 0) = 1 - \Phi\left(\frac{x_i'\beta}{\sigma}\right)}$$

(b)

(i) maximum likelihood : given we have observed $x_1, \dots, x_n$ and $y_1, \dots, y_n$, what is the probability that the distribution parameters are in fact β and σ

[that is, what are the most likely parameters β and σ given the assumed distribution and observed data]

(ii)

$$f_{\beta,\sigma} \cancel{(y_i|x_i)} (y_i|x_i) = \left(\Phi\left(\frac{x_i'\beta}{\sigma}\right)\right)^{y_i}\left(1-\Phi\left(\frac{x_i'\beta}{\sigma}\right)\right)^{1-y_i}$$

$$L_{y_1,\dots,y_n,x_1,\dots,x_n}(\beta,\sigma) = \prod_{i=0}^{n}\left[\Phi\left(\frac{x_i'\beta}{\sigma}\right)\right]^{y_i}\left[1-\Phi\left(\frac{x_i'\beta}{\sigma}\right)\right]^{1-y_i}$$

$$l_{y_1,\dots,y_1,x_1,\dots,x_n}(\beta,\sigma) = \sum_{i=0}^{n}\left\{y_i \ln\left(\Phi\left(\frac{x_i'\beta}{\sigma}\right)\right)+(1-y_i)\ln\left(1-\Phi\left(\frac{x_i'\beta}{\sigma}\right)\right)\right\}$$

Likelihood function <u>is</u> the joint distribution function of $(y_1, \dots, y_n)$ given $(x_1, \dots, x_n)$.

log-likelihood = ln (likelihood function)

Likelihood = joint distribution = $\prod_{i=0} f(y_i|x_i)$

(for iid draws)

(c)

$\sigma = 1$

$$l_{y_1,\dots,y_n,x_1,\dots,x_n}(\beta) = \sum_{i=0}^{n} y_i \ln(\Phi(x_i'\beta)) + (1-y_i)\ln(1-\Phi(x_i'\beta))$$

$$\frac{\partial l_{y_1,y_n,x_1,x_n}(\beta)}{\partial \beta} = \sum_{i=0}^{n} y_i \frac{x_i'\phi(x_i'\beta)}{\Phi(x_i'\beta)} + (1-y_i)\frac{-x_i'\phi(x_i'\beta)}{1-\Phi(x_i'\beta)}$$

$\phi(x)$: normal density

$\Phi(x)$: normal CDF

$$\frac{\partial \Phi(x)}{\partial x} = \phi(x) \quad \text{by definition.}$$

No closed form solution $\Rightarrow$ approximate function by Gauss-Newton method to find solution.

gg_i

(d)

(i) $$\mathbb{E}\left[\frac{\partial^2 l(\beta,x)}{\partial\beta^2} + \frac{\partial l(\beta,x)}{\partial\beta}\frac{\partial l(\beta,x)}{\partial\beta'}\right] = \int \frac{1}{f(x,\beta)}\frac{\partial^2 f(x,\beta)}{\partial\beta^2} f(x,\beta)\,dx$$

$$= \int \frac{\partial^2 f(x,\beta)}{\partial\beta^2}\,dx$$

$$= \frac{\partial^2}{\partial\beta^2}\int f(x,\beta)\,dx$$

$$= \frac{\partial^2}{\partial\beta^2}\cdot 1$$

$$= 0$$

hence

$$\mathbb{E}\left[\frac{\partial^2 l(\beta,x)}{\partial\beta^2}\right] = -\,\mathbb{E}\left[\frac{\partial l(\beta,x)}{\partial\beta}\frac{\partial l(\beta,x)}{\partial\beta'}\right]$$

(ii) MLE are consistent & asymptotically normal.

(iii) ?

(4)

(a)

(i) $\ln Y_i = \ln A_R + \alpha_R \ln K_i + \beta_R \ln h_i + \ln u_i$

$\ln Y_i = \ln A_p + \alpha_p \ln K_i + \beta_p \ln h_i + \ln u_i$

let $\tilde{Z}_i = \ln Z_i$

model : $\tilde{Y}_i = \cancel{\beta} A_p + \alpha_p \ln K_i + \beta_p \ln h_i$

$+ A_\Delta D_i + \alpha_\Delta D_i \ln(K_i) + \beta_\Delta D_i \ln(h_i) + \ln u_i$

$D_i = 0$ (Poor)
and we have $\tilde{Y}_i = A_p + \alpha_p \ln K_i + \beta_p \ln h_i + \ln u_i$

$D_i = 1$ (Rich)
and we have : $\tilde{Y}_i = (A_p + A_\Delta) + (\alpha_p + \alpha_\Delta) \ln K_i + (\beta_p + \beta_\Delta) \ln h_i + \ln u_i$

(ii)
take logs.

Poor: $\ln \hat{A}_p = \hat{A}_p$, $\hat{\alpha}_p = \hat{\alpha}_p$, $\hat{\beta}_p = \hat{\beta}_p$

Rich: $\ln \hat{A}_R = \hat{A}_p + \hat{A}_\Delta$, $\hat{\alpha}_R = \hat{\alpha}_p + \hat{\alpha}_\Delta$, $\hat{\beta}_R = \hat{\beta}_p + \hat{\beta}_\Delta$

(iii)
log-log : elasticities

$\hat{\alpha}_p$, $\hat{\beta}_p$ are $\% \Delta$ in output for $\% \Delta$ in K / L
(Poor countries)

$(\hat{\alpha}_p + \hat{\alpha}_\Delta)$, $(\hat{\beta}_p + \hat{\beta}_\Delta)$ are $\% \Delta$ in output for $\% \Delta$ in K/L
Rich countries.

$$\frac{d \ln y}{d \ln x} = \frac{dy}{dx} \frac{x}{y}.$$